PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 26 December 15, 1940 Number 12

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THE DEVELOPMENT OF MELANOPHORES FROM EMBRYONIC MOUSE TISSUES GROWN IN THE COELOM OF CHICK EMBRYOS

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Communicated October 31, 1940

Transplantation experiments have demonstrated conclusively that migratory pigment-forming cells originate from the neural crest in amphibians[3] and in birds;[2,4,11] but so far no attempt has been made to determine whether or not the same is true in mammals. In their studies on the origin of pigment in birds Eastlick[4] and Ris[11] found the method of grafting embryonic tissue to the coelomic cavity of a developing chick embryo as used by Hamburger[5] particularly suitable. As far as the author is aware, tissues other than those of birds have not been grown in the coelom, but it is known that embryonic tissue of both the rat[7,9] and the mouse[10] will differentiate to a certain extent on the chorio-allantoic membrane of the chick. The possibility that the intracoelomic grafting method might be used with success for testing the origin of pigment cells in a mammal seemed too tempting to leave untried, therefore the following experiments were undertaken.

Experimental Procedure.—The mice from which the donor embryos were taken were of a strain known to produce only black offspring.[14] Embryos were used at ages ranging from $8^1/_2$ to 12 days, timed from the observance of the vaginal plug. They were removed from the uterus (under anesthesia) one at a time as needed. Those not used were fixed as controls for histological study. Each donor embryo was placed in warm sterile Locke's solution in a small Petri dish placed under a binocular dissecting microscope. After all membranes had been removed and the somites counted, small pieces (0.5 to 1 mm. in length) of skin ectoderm, plus some of the mesoderm lying directly beneath it, were isolated from various levels along the antero-posterior axis and implanted one at a time into the coelomic cavity of White Leghorn host embryos of 60 to 70 hours' incubation. White Leghorns were chosen as hosts for these experiments because of the fact

that they do not regularly exhibit pigment in their coelomic epithelium. Only 10 individuals out of 115 examined especially for this purpose on the 18th or 19th days of incubation showed even a trace of pigment. When present the pigment was more or less evenly distributed in the coelomic epithelium with greatest concentration in the region of the umbilicus. Examination of the pigmented region shows that nearly all of the melanophores had undergone degeneration, i.e., had retracted their processes, and persisted only as small dense balls of pigment (Fig. 5). For an account of similar degeneration in regenerated feathers and *in vitro* cultures see Hamilton.[6]

When the donors had well-developed limb buds (at 11 and 12 days), skin ectoderm plus adhering mesoderm from these regions as well as that from the head and trunk was tested. In the younger stages, however, the entire limb bud or longitudinal halves of limb buds were grafted, as were also pieces of somite and lateral plate with and without the neural tube. Aside from the intracoelomic implantations, a few grafts were made to the base of the developing wing bud. For this purpose skin ectoderm as free as possible of adhering mesenchyme was taken from the posterior head region of 10-day embryos (about 30 somites).

For making the isolations, steel beading needles with very fine points, sealed into glass tubing for handles, and a Bowman's iris knife were used. To facilitate handling, the pieces were stained by touching them lightly, before removal from the body of the embryo, with small glass rods having agar-coated tips stained with Nile blue sulphate.

The preparation of the host embryo was essentially that described previously for transplanting embryonic chick tissue to the body wall or to the limb bud.[13] A small rectangular piece of shell approximately 1×0.5 cm. was sawn and lifted out immediately over the body of the embryo whose position had been previously determined by candling. The shell membrane was picked away carefully under a binocular microscope, thus exposing the embryo lying on the yolk beneath. By touching the embryonic area with the stained tips of the above-mentioned glass rods, the membranes become visible immediately and can be very easily opened. Usually at the stages used the amnion had not grown over the posterior regions, so did not have to be dealt with. When the body wall was thus exposed a small slit was made with a glass needle in the ectoderm and somatic mesoderm just anterior to the leg bud region of either the right or left sides. The implant of donor tissue was transferred through the window to the host blastoderm in a capillary pipette of suitable size with a small amount of saline solution. It was pushed through the slit gently with a glass needle into the coelomic cavity. The original piece of shell was fitted into the window the edges sealed with melted paraffin and the egg returned to the incubator. During the operation and afterwards the egg was placed on a cotton pad in a

Syracuse watchglass to prevent rotation. The host embryo was removed and dissected usually on the 19th day of incubation. A thorough examination of the coelom was made for the graft and melanophores migrating from it. In all positive cases the grafts were fixed *in situ* in Bouin's fluid for histological study.

Observations.—When the hosts were dissected, it was found that the implanted mouse tissue had differentiated into a definite body or graft in 62 out of 101 cases. Except when entire limb buds or halves of limb buds had been taken originally, the grafts were small, rounded or oval-shaped bodies not exceeding 4 mm. in diameter, well vascularized and firmly attached to the wall of the coelom in the posterior region of the body. Often the free surface of the graft was connected to mesenteries, intestine, colacal wall or even to some other part of the coelomic epithelium by strands of connective tissue. Always the graft surface was covered with a thin, smooth, transparent layer of epithelium. The great majority of grafts contained well-developed hairs, very conspicuous when pigmented (Fig. 1), less so when unpigmented. Due to the fact that the implanted piece of skin ectoderm always tends to round up into a hollow ball or vesicle, the developing hairs are most often directed inwards towards the center of the graft.

The total age of the graft tissue when removed for study is equivalent to that of 3- to 7-day postnatal mice ($8^1/_2$ to 12 days + approximately 16 days in the coelom) and of strictly comparable differentiation. At these ages pigmented hairs can be seen in the normal mouse and with low magnification numerous melanophores scattered in the skin between the hairs are easily visible. Even at birth melanophores concentrated in the developing hair follicles and distributed in the skin between them can be found in the dorsal and lateral regions of the body. In 34 cases specific mouse melanophores densely crowded with coarse black melanin granules were found in the graft or migrating in the coelomic epithelium of the host in its immediate vicinity (Figs. 2, 3 and 4).

The larger size of the mouse melanophore as compared with that of the White Leghorn, and its extreme blackness make it very conspicuous and easy to detect in the coelomic epithelium. Usually there are two diametrically opposed processes ranging from 50 to 100 micra in length (Fig. 3). Occasionally several cells hang together and form long chains (Fig. 4).

Whether or not melanophores were produced by the grafted tissue was soon found to depend upon the age and the region of the embryo from which the implant was taken. Differences in these respects are indicated in the following paragraphs, grafts from older embryos being described first.

Twelve-Day Donors.—Skin ectoderm plus adhering mesoderm from all regions tested, namely, neck, shoulder, rump and limb buds, gave 15

successful grafts out of 20 transplants. All had well-developed pigmented hairs, many approximately 1 mm. long (Fig. 1). Histological study of one typical graft showed a vesicular arrangement of well-differentiated skin with numerous hairs growing out into a central cavity. Melanophores were seen in the subcutaneous tissue and dermal layer of the skin as well as in the hair follicles themselves. Although all of the grafts contained pigment, only 9 of the 15 showed migration of melanophores into the surrounding coelomic epithelium of the host and in none was it very extensive.[15]

Eleven-Day Donors (40–45 Somites).—Eleven grafts were recovered from 22 hosts examined. As in the preceding series, skin ectoderm and mesoderm from the head and trunk levels produced hairs with pigment, and in 5 of the 7 cases rather extensive migration of mouse melanophores into the host coelomic epithelium was observed. Unlike the preceding, however, skin ectoderm and mesoderm from the limb buds (4 cases, 2 examined histologically) gave well-differentiated hair follicles wholly without pigment: no melanophores were found either in the graft or in the host coelomic epithelium.

Ten to 10½-Day Donors (28–35 Somites).—In addition to transplanting

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DESCRIPTION OF PLATE 1

Figure 1. Graft developed from small piece of skin ectoderm + adhering mesenchyme (0.8 × 0.5 mm.) from trunk at base of right fore limb bud of a 12-day mouse embryo, grown 16½ days in the embryonic coelom of a White Leghorn chick. Note mass of well-developed hairs fully pigmented. Photographed *in situ*. (× 10.)

Figure 2. Portion of coelom of White Leghorn host embryo (19¼ days) showing migration of mouse melanophores from graft (*G*). Note concentration of melanophores along walls of small blood vessels. Produced by implanting right half of neural tube and adjacent somite material at 25th somite level of 10-day (31 somites) mouse embryo to 70-hr. chick coelom. (× 10.)

Figure 3. Unstained whole mount of portion of coelomic epithelium of White Leghorn host embryo showing migrating mouse melanophores produced from implant of hind limb bud skin ectoderm (0.7 × 0.3 mm.) from 12-day mouse embryo, grown 16½ days in embryonic chick coelom. Graft (not shown) contained hairs fully pigmented. (× 85.)

Figure 4. Unstained whole mount of portion of coelomic epithelium of a White Leghorn host embryo showing migrating mouse melanophores produced from dorsal trunk skin ectoderm (0.7 × 0.4 mm.) of 45-somite (11-day) embryo, grown 17½ days in coelom. Note melanophores hanging together in long chains. (× 85.)

Figure 5. Unstained whole mount of portion of coelomic epithelium of a normal 19-day White Leghorn embryo showing degenerated melanophores with retracted processes. (× 85.)

Figure 6. Portion of wing of 17-day White Leghorn host embryo showing dense mass of mouse melanophores in the skin and subcutaneous tissue. Produced by implanting head skin ectoderm (1 × 0.4 mm.) from a 10-day (30-somite) mouse embryo to the wing bud region at 60 hrs. of incubation. (× 25.)

Figure 7. Same as figure 6 with greater magnification showing mouse melanophores at the periphery of mass. (× 85.)

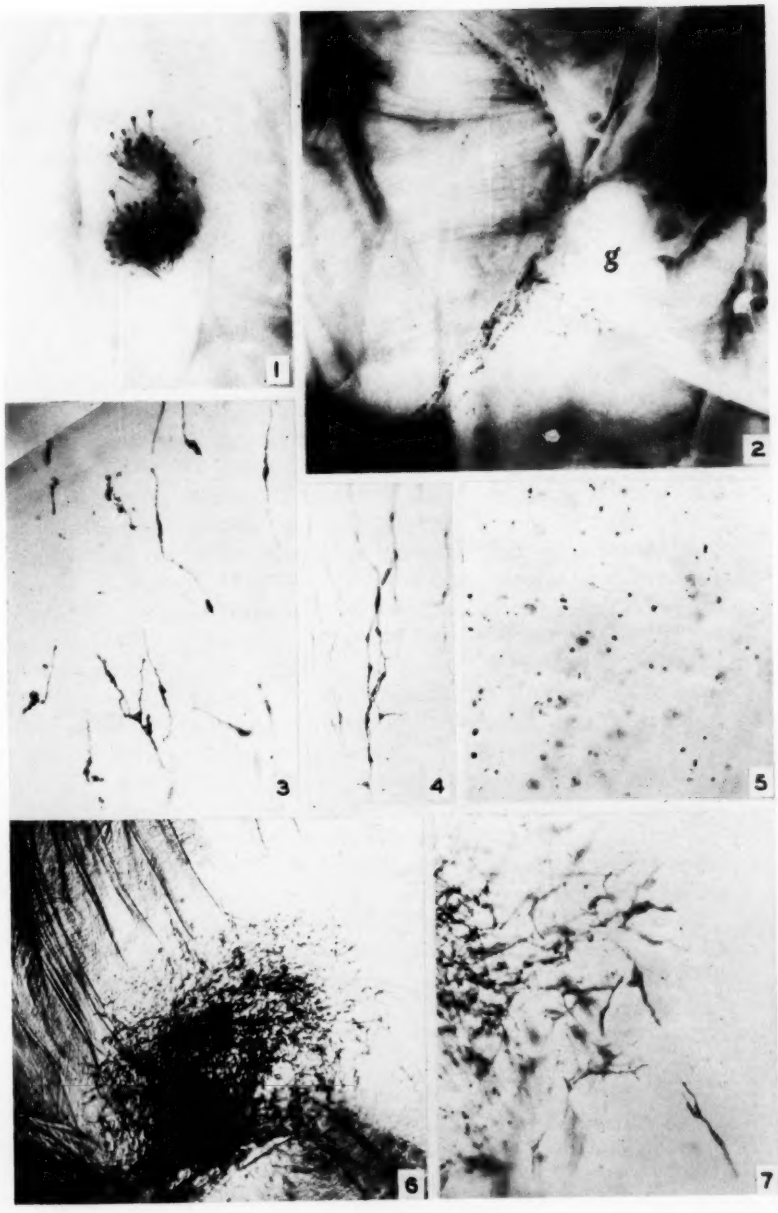

PLATE 1

skin ectoderm and subjacent mesoderm from various levels as done in the preceding series, grafts were made also of entire limb buds, longitudinal halves of limb buds, short segments of somite and somite plus the adjacent half of the neural tube. Thirty grafts were obtained from 40 hosts examined; 6 were studied histologically.

Head skin and trunk skin from the dorsal region, i.e., directly over the neural tube, at all levels produced melanophores both in the graft and in the coelomic epithelium about the graft. But no melanophores developed from either of the limb buds when grafted in their entirety or in part, although well-differentiated hair follicles were abundant. The entire limb bud (12 cases) showed a remarkably normal differentiation of skeletal parts, muscle, skin and hair.

The most beautiful and extensive migration of mouse melanophores into the coelomic epithelium came from grafts of somite including the neural tube taken from the leg bud level or just anterior to it. In the best of the six cases obtained, melanophores in large numbers had migrated for a distance of about 5 mm. in two directions along the walls of small blood vessels (Fig. 2). The grafts themselves showed no pigment either macroscopically or microscopically. In the two studied histologically small skin vesicles with well-differentiated skin were found but no hair follicles. The bulk of the graft consisted of a mass of central nervous tissue (spinal cord) showing strikingly normal differentiation.

Similar grafts obtained from the intact somite not including the neural tube showed no pigment either in the graft or in the coelomic epithelium (6 cases). Histological examination of two of these showed good development of skin and hair follicles. One in addition contained skeletal muscle and small cartilages.

In view of the fact that the developing wing bud of the chick has proved such a favorable site for melanophore migration in birds in general, it seemed desirable to make the same test with mouse melanophores. Thus small pieces of head skin ectoderm relatively free of adhering mesoderm were grafted to the wing bud region of White Leghorn host embryos of approximately 60 hours' incubation. In one case out of three, examination on the 17th day showed a small black spot 2 mm. in diameter, in the skin and subcutaneous tissue of the wing of the host at the implantation site (Fig. 6). When examined under a microscope the black area was found to be densely crowded with large branching mouse melanophores exactly like the normal and like those developed in the coelom (Fig. 7). None of the melanophores had migrated into the overlying feathers.

Eight and One-half to 9-Day Donors (15–20 Somites).—Embryos at this stage had not developed limb buds and were very fragile. Two types of transplants were made: those in which the somites and lateral plate were cut entirely free of the neural tube, and those in which the neural tube was

left attached. Levels from the 10th somite posteriorly were thus tested. Six grafts were recovered from eight hosts alive at the 19th day of incubation; two came from implants of pure somite and lateral plate and produced no pigment; the other four from implants of somite and lateral plate including neural tube. In none of the four containing neural tube was pigment found in the graft itself, but two showed typical mouse melanophores migrating in the coelomic epithelium around the graft site. In one, the melanophore migration was particularly pretty and extended for several millimeters along a strand of tissue attaching the graft to the cloacal wall. Even though the number of grafts analyzed in this series is small, the results are definitely in line with the others.

Discussion.—The results summarized above show that between $8^1/_2$ and 12 days of gestation the capacity of the grafted embryonic tissue to produce melanophores spreads rapidly both antero-posteriorly and medio-laterally. Grafts from the head region at all stages tested produced pigmented hairs consistently (no data on head skin of $8^1/_2$-day embryos), but whether or not grafts from the somites and limb buds produced hairs with pigment depended upon both age and body level. For example, at $8^1/_2$ days no melanophores developed from grafts of somite material when completely isolated from the neural tube; at 10 days skin ectoderm and mesoderm from somites of the anterior trunk levels including the fore limb region gave hairs fully pigmented while similar grafts from the hind limb region gave hairs entirely without pigment; at 11 days somite grafts from all levels including the posterior limb region produced hairs with pigment. Ten and 11-day limb buds, entire or in part, failed to develop pigment, although skin and hair follicles were abundant. By 12 days, however, pigmented hairs developed regularly from implants of skin + adhering mesenchyme from both limb buds.

This occurrence of well-differentiated hairs wholly without pigment from certain body regions at certain developmental stages suggests strongly that the melanophores migrate into the developing hair follicles from an outside source. That this source lies in or is closely associated with the neural tube is demonstrated by the fact that at those stages in which somite material failed to produce pigment cells, as in the hind limb region of a 10-day embryo, for example, the same regions would develop pigment if the implant was made to include the adjacent part of the neural tube.

The results of the present experiments parallel very closely those obtained from similar transplantation experiments in the chick. By grafting limb buds from chick embryos (24–30 somites) of pigmented breeds to the coelom of White Leghorn hosts of similar ages, Eastlick[4] was able to show that white feathers only were produced unless the implant included body wall material (skin ectoderm and mesoderm) up to the neural tube. Ris[11] obtained similar results by transplanting to the chorio-allantois as well as

to the coelom. Limb buds from embryos of pigmented breeds isolated at 72 hrs. developed white feathers at both sites while the same isolations later, i.e., at 90 hrs. or more, produced pigmented feathers in the graft.

Further evidence for the extra-epidermal origin of pigment cells in the chick, and their lateral migration from the neural tube region into the overlying ectoderm and into the limb buds, is obtained from the transplantation experiments of Willier and Rawles,[13] and Watterson.[12] Small pieces of skin ectoderm or pure mesoderm from embryos of one breed were implanted into the developing wing bud region of host embryos of a similar age (67–108 hrs.) but of a genetically different breed. Implants from the head or trunk regions at all ages tested produced an area of donor-colored feathers on the host at and about the site of implantation. But implants from the wing bud of embryos younger than 80 hours or from the leg bud earlier than 96 hours failed to produce pigment in the host feathers.

Using the same method of implantation to the limb bud, Dorris[2] obtained areas of pigmented skin and feathers in white hosts by implanting thin strips of the rising neural folds (neural crest region) anterior to the first somite from embryos of 3–10 somites of black breeds. Ris[11] correlated the development of pigment in grafts with the morphological appearance of the neural crest at the time of isolation. By transplanting portions of embryos (potentially pigmented) from various levels at successive stages in their development to the embryonic coelom of White Leghorns, he was able to show quite convincingly that the inclusion of the neural crest, migrating cells of the crest or presumptive neural crest, in the transplant was essential for the development of melanophores.

The similarity of the intra-coelomic grafts of mouse tissue with those obtained by Ris from chick and other bird implants is indeed striking. While the mouse melanophores do not migrate nearly as extensively in the White Leghorn coelomic epithelium as those from birds, they do nevertheless tend to wander out from the grafts and follow blood vessels, nerves and strands of connective tissue in much the same way. The failure of mouse melanophores to migrate to any extent in the wing bud, as observed in the one case obtained, is in contrast with the tremendous migration exhibited by bird melanophores in the same site.

The extensive work of Holmdahl[8] on the origin and development of the neural crest in birds and mammals shows how very much alike these two groups are in respect to this structure. In both it arises first in the region of the mid-brain at early somite stages, and develops posteriorly along the neural tube, migrating laterally as the age of the embryo increases. Histological examination of control mouse embryos at the stages and levels used for transplantation shows that the regions which produced pigment in the grafts coincide with the regions from which the neural crest had already migrated from the neural tube. At stages where in posterior levels

the crest had not migrated, or had not migrated sufficiently far to be included in the transplant ($8^1/_2$–10 days), no pigment resulted. Such facts strongly suggest that the same source and mode of migration of melanophores obtains in the mouse as in the bird.

In summary, the experimental as well as the morphological evidence shows that in the mouse, as in the chick, the melanophores arise from an outside source, the neural crest, presumably, and migrate into the skin and developing hair follicles. Further, it is demonstrated that the embryonic coelom of the chick is an excellent site for the growth and differentiation of implanted embryonic mammalian (mouse) tissue.

[1] The experiments upon which this report is based were carried out at the University of Rochester. The manuscript was prepared at Stanford University during a stay of four months in the laboratory of Professor C. H. Danforth. I am deeply indebted to him for the interest he showed in examining the material and for many helpful suggestions. My appreciative thanks are also due Professor B. H. Willier for a critical reading of the manuscript.

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[11] Ris, Hans, *Physiol. Zoöl.* (in press) (1941).

[12] Watterson, R. L., *Anat. Rec.*, **70,** Supp. 4, 100 (1938).

[13] Willier, B. H., and Rawles, M. E., *Physiol. Zoöl.*, **13,** 177–199 (1940).

[14] Heterozygous black females were crossed with homozygous black males (C57 black Bar Harbor strain) and the F_1 females from this mating back-crossed to the male parent for successive generations.

[15] A few trial cultures of skin ectoderm were grown *in vitro* in a medium of chick embryonic extract and plasma by Mr. Howard L. Hamilton. Both hair follicles and melanophores differentiated in the explants.

GALACTIC AND EXTRAGALACTIC STUDIES, XI. NOTE ON THE PERIOD FREQUENCY OF GALACTIC CEPHEIDS

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Communicated November 14, 1940

1. Introduction.—The striking correlation of the period lengths of classical Cepheids with the density of the star fields in which they lie was described for the Small Magellanic Cloud in the fifth paper of this series.[1] The large number of variables for which periods had been determined permitted the examination of the frequency of periods separately for various sections of the system. Both the mean period and the median period show the dependence on distance from the nucleus of the Cloud, the longer-period stars being preferentially in the region of high star density and high gravitational potential.*

The correlation is neither strongly confirmed nor denied by the available material for the periods of Cepheids in the Large Magellanic Cloud; further work is in progress to provide a competent test. The distribution of the periods of Cepheids in our own Galaxy was reëxamined some time ago, but little weight could be given the available evidence because of the strong selection factors that operate against a homogeneous collection of material. Irregular and undetermined amounts of space absorption make it impossible to determine the true magnitudes and distances of the faint galactic Cepheids; and, furthermore, the discovery surveys are complete only in limited areas, and there are complications with the boundaries of the Galaxy in the anticenter direction and in high latitudes. These boundary questions arise, of course, from the interdependence of period and intrinsic luminosity.

The problem of the distribution of galactic Cepheids is now taken up again because many new periods have been obtained recently for faint stars in the direction of the galactic center. From this new material strong support is obtained for the hypothesis that the period-field density correlation found in the Small Magellanic Cloud may be a general property of Cepheid variation.

2. Faint Classical Cepheids near the Galactic Center.—In various earlier papers Miss Swope and I have reported on the variable stars in certain fields in Sagittarius, Scorpio and Ophiuchus, in the general direction of the central nucleus of the Milky Way.[2] Long-period variables appear in the fields most numerously, but cluster-type Cepheids and eclipsing stars are also abundant, and this region is the "home" of the faint galactic novae. The peculiar distribution of the various types reflects on the absolute magnitudes of the different kinds of variables and also on the distribution of

obscuring matter in this section of the Milky Way. Classical Cepheids are, however, relatively scarce in these great southern star clouds. In fact, the relative frequency of cluster-type and classical Cepheids in the direction of the galactic nucleus is much the same as it is in many of the globular star clusters, where there are scores of cluster-type variables and only two or three Cepheids of long period.

Probably obscuration prevents our low-latitude survey from reaching into the regions of the Milky Way beyond the central nucleus. The variables that are found by the thousands in low latitudes throughout this fourth quadrant of longitude (270° to 360°) are either in the rich nuclear star clouds or between them and the Sun.

Variable-star surveys have been carried through with reasonable completeness to magnitude 16.5 (or occasionally fainter with plates made with the Bruce refractor) in the following Milky Way fields, for which the approximate galactic coördinates of the centers are given:

MWF	LONGITUDE, °	LATITUDE, °
184	315	−7.5
185	322.5	+7.5
186	322.5	0.0
187	322.5	−7.5
189	330	0.0

In table 1 the names, galactic coördinates, median magnitudes and periods are given for the classical Cepheids in these five fields. Of the thirteen stars in MWF 184 only CQ Scorpii has been published, but the others will appear in a forthcoming *Harvard Observatory Bulletin*, with magnitudes, light elements and equatorial coördinates derived by Miss Constance Boyd, who discovered eleven of the thirteen variables. The light curves of all thirty-three variables have been determined from Harvard plates. Four of them, AL CrA, V532 Sgr, V557 Sgr, IU Sco, have unusual curves, and the periods of the second and fourth are the longest in the whole list; but probably all are classical Cepheids.

Since the direction to the galactic center is very near to longitude 327°, latitude 0°, it appears that the five fields here under consideration, which cover a total of about two hundred and eighty square degrees, are all closely involved with the galactic nucleus.† The positions are shown on the accompanying small-scale photograph of the nuclear region (Fig. 1). On this photograph the boundaries of the fields have been roughly sketched in and also lines indicating the galactic coördinates. The small number of classical Cepheids in MWF 186 presumably results from the heavy obscuration along the galactic equator.

MWF 184 is the field farthest from the center. None of its classical Cepheids is in the corner of the field toward the galactic circle, apparently

because heavy intervening obscuration affects that region. The long period variables for this field have been published[3] and show a similar avoidance of the obscured region. In figure 2 the positions of the long-period variables are indicated by dots and the positions of the classical Cepheids by circles.

TABLE 1

THIRTY-THREE CLASSICAL CEPHEIDS IN MWF 184, 185, 186, 187, 189

NAME	FIELD	λ	β	$\dot{m}$	PERIOD d
X Sgr	189	329°	− 1°	5.35	7.012
W Sgr	189	329	− 5	5.4	7.59
BF Oph	185	325	+ 8	8.2	4.068
HV 7902	186	322	− 1	8.5	4.528
RY Sco	187	324	− 5	8.95	20.31
HV 10513	184	314	−11	12.2	19.818
V626 Sgr	187	325	− 8	12.35	26.762
AL CrA	187	323	−11	12.5	17.061
AV Sgr	189	335	− 2	12.6	15.411
ET Oph	185	324	+ 8	12.8	25.22:
HV 7885	186	317	− 1	12.95	16.263
V564 Sgr	187	325	− 8	13.1	27.8
CQ Sco	184	319	− 7	13.55	30.412
HV 10260	189	333	+ 3	13.55	14.865
HV 10460	184	315	− 5	13.7	11.155
CE Oph	185	324	+10	13.95	15.89
V446 Sco	187	322	+ 3	14.0	28.634
HV 10302	189	331	− 2	14.15	5.748
V532 Sgr	187	323	− 7	14.35	34.21
HV 10495	184	317	− 8	14.4	15.234
HV 10469	184	313	− 8	14.4	1.176
HV 10488	184	315	− 8	14.45	24.693
HV 10456	184	312	− 6	14.5	11.821
HV 10505	184	319	− 8	14.75	16.547
HV 10246	189	331	+ 3	14.8	29.533
V557 Sgr	187	323	− 8	14.8	16.260
HV 10512	184	318	− 9	15.1	15.632
HV 10467	184	317	− 5	15.2	15.66
HQ CrA	187	322	−11	15.4	1.415
HV 10485	184	318	− 7	15.5	15.526
IU Sco	185	320	+ 6	15.55	69.05
HV 10484	184	319	− 7	15.55	24.09
HV 10509	184	317	− 9	16.0	2.441

In none of the five fields have we found external galaxies, bright or faint.[4] We must assume, therefore, that the total photographic absorption may be several magnitudes in the most obscured regions and perhaps two magnitudes in the richest star fields that are without visible dark lanes.

3. *The Abnormal Frequency of Periods.*—From table 1 it is seen that the

frequency of periods is unlike that found elsewhere, even in the nucleus of the Small Magellanic Cloud. The mean period for the thirty-three stars is 17.94 days, and the median value is 15.99 days. Dropping the extreme value 69.05 days for IU Scorpii, we have the mean and median periods 16.34 and 15.78 days, respectively. Figure 3 illustrates this unusual preference for long periods, with the remarkable clustering at sixteen days. The median apparent brightness for the thirty-three stars is 14.0; for the eleven with periods between 14.8 and 17.1 days it is 14.4.

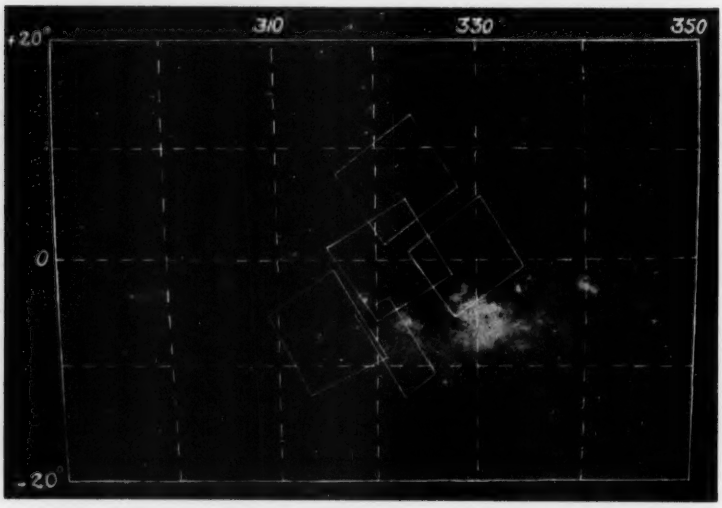

FIGURE 1

Composite photograph of the galactic nucleus showing location of Milky Way fields. Only one-third of MWF 185 has been thoroughly studied. The ordinates are galactic latitudes; the abscissae, galactic longitudes. The center is at 0°, 327°, approximately.

The question arises at once whether or not some factor of selection is responsible in this survey for the infrequence of Cepheids with periods from three to seven days—the interval that includes 51 per cent of the 340 classical galactic Cepheids now known and catalogued. Many cluster-type and eclipsing variable stars have been found that are more than a magnitude fainter than the Cepheids with sixteen-day periods; and since Cepheids of five-day period are intrinsically less than a magnitude fainter, these ordinary Cepheids should have readily been discovered if present. It is necessary to conclude, therefore, that the scarcity is real and that very few Cepheids of ordinary period exist in the fields around the galactic center.

The situation in MWF 184 is specially worth noting. The median mag-

nitude of the thirteen classical Cepheids is 14.5, and the median period is 15.6 days. Only two periods are shorter than eleven days, whereas, in the same field, intermingled with these classical Cepheids, are many cluster-type Cepheids with median magnitudes from 15.0 to 16.1. Since the classical Cepheid with a period of a few days is, when present, an easy variable to discover on the Harvard photographs, we must conclude that such stars are essentially absent from MWF 184.

The total material is not large but appears quite sufficient to demonstrate that the classical Cepheids in the vicinity of the galactic center show the same tendency to high luminosity, large mass and long period, as is shown by the classical Cepheids in the center of the Small Magellanic Cloud.

Because of uncertainty as to the total amount of intervening space absorption, we cannot be sure that these long-period Cepheids are actually as distant as the galactic nucleus, that is, nine or ten kiloparsecs distant. But the observed differences between the average magnitudes of the long-period variables, cluster-type Cepheids and classical Cepheids in the same fields are consistent with that assumption.

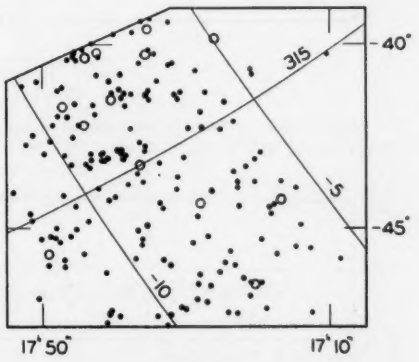

FIGURE 2

Distribution of long-period variables (dots) and classical Cepheids (open circles) in MWF 184. Galactic coördinates are labelled inside the figure; equatorial coördinates on the outside.

The distance modulus, $m' - M$, corresponding for MWF 184 to the median observed magnitude 14.5 and the median period 15.6 days, is 16.85. To justify the assumption that these long-period Cepheids are closely associated with the galactic nucleus, and therefore at a distance of approximately ten kiloparsecs, we would require for this "median Cepheid" a space absorption of 16.85 − 15.0 = 1.85 magnitudes. This is perhaps an acceptable value for the bright star clouds, although much larger absorption is probable in the adjacent rifts along the galactic equator.

4. *Galactic Longitude and Period Length.*—In the 1940 catalog of variable stars, issued from the Berlin-Babelsberg Observatory, there are 336 variables listed as classical galactic Cepheids with galactic latitudes lower than $\pm 30°$ (only three now listed are in higher latitudes, except for a number in the vicinity of the Magellanic Clouds that actually belong to those systems and not to our own Galaxy). The distribution of these 336 vari-

ables in thirty-degree intervals of galactic longitude is shown in table 2, which includes also for each group the median period, $\dot{P}$, and the mean

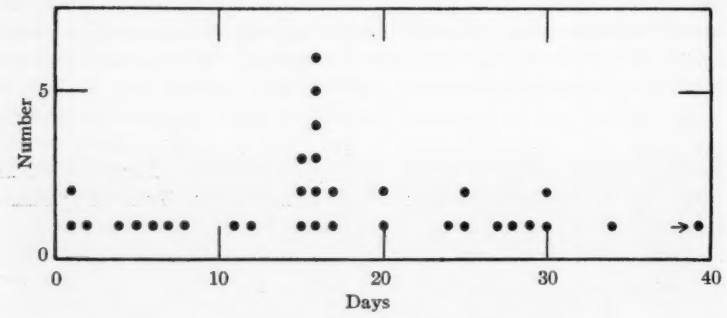

FIGURE 3

Distribution of periods (to nearest whole day) for the thirty-three classical Cepheids in MWF 184, 185, 186, 187, 189. The long period is 69.05 days.

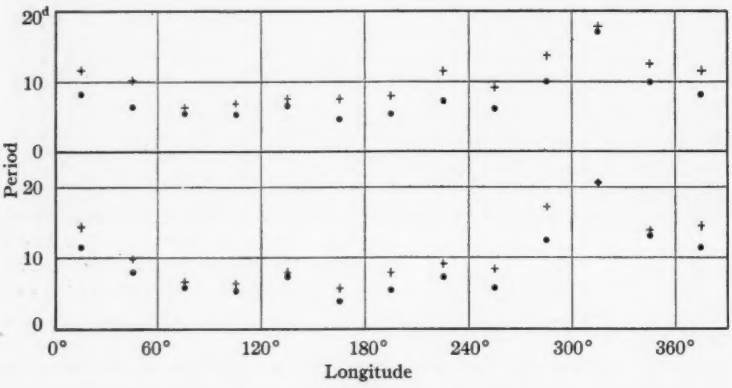

FIGURE 4

Dependence of period on galactic longitude; above, for the 336 known galactic Cepheids (periods $>1^d$; latitudes $< \pm 30°$), of all magnitudes (table 2); below, for the 227 galactic Cepheids with photographic magnitudes fainter than 10.0 (table 3). Crosses indicate mean periods and dots median periods in groups for 30° intervals in longitude.

value of the period, $\overline{P}$. In figure 4 the values of the median and mean periods are plotted.

Since sixty-eight per cent of the classical galactic Cepheids are fainter than the tenth magnitude, the material is given separately in table 3 for

these fainter and more distant variables; and the relation of period to longitude for them is plotted in the lower part of figure 4.

Whether we consider means or medians, consider all the variables or the fainter variables only, the conclusion is the same: in the interval of longitude that includes the galactic nucleus, say from 270° to 360°, the periods of the classical Cepheids average longer than in any other longitude. At first glance this result seems to be a clear confirmation of that already found from the study of the five fields near the galactic center, and of the result found for the Small Magellanic Cloud; but, in view of the numerous selection factors involved in the discovery and study of galactic Cepheids, the apparent confirmation should not be too strongly emphasized. It is probable that the scarcity of the longest-period Cepheids in the anticenter

TABLE 2

LONGITUDE-DISTRIBUTION OF GALACTIC CEPHEIDS

(All Magnitudes; Periods in Days)

λ	15°	45°	75°	105°	135°	165°	195°	225°	255°	285°	315°	345°
Number	22	27	22	18	16	21	26	17	76	34	21	36
$\dot{P}$	8.185	6.32	5.405	5.325	6.61	4.66	5.385	7.19	6.18	10.05	17.06	9.945
$\overline{P}$	11.49	10.15	6.28	6.76	7.57	7.59	7.97	11.51	9.15	13.70	17.89	12.59

TABLE 3

LONGITUDE-DISTRIBUTION OF GALACTIC CEPHEIDS

(Magnitudes Fainter than 10.0; Periods in Days)

λ	15°	45°	75°	105°	135°	165°	195°	225°	255°	285°	315°	345°
Number	13	19	14	14	12	15	21	6	50	21	16	26
$\dot{P}$	11.51	7.86	5.87	5.325	7.395	3.89	5.73	7.355	6.005	12.64	20.71	13.23
$\overline{P}$	14.51	9.89	6.67	6.53	8.01	5.83	7.91	9.30	8.51	17.30	20.68	14.02

region cannot be attributed solely to the nearness of the observer to the boundary of the Milky Way in that direction. The possibility however must be kept in mind that perhaps a part of the difference between center and anticenter is a matter of depth available for survey. But there should be little "boundary" selection in longitudes 60° and 240°, at right angles to the direction to the center; and we find in these longitudes a similar infrequence of the peculiarly long-period Cepheids such as predominate in MWF 184 and adjacent regions.

5. *Summary.*—The material presented in the tables and figures of the present note appear to confirm a phenomenon previously discovered in the Small Magellanic Cloud: in regions of highest star density the periods of classical Cepheids are distinctly longer on the average than at the borders of the system or in regions of intermediate star density.

The evidence now presented for the galactic Cepheids includes results

obtained from five rich variable star fields at or near the galactic center. In these regions classical Cepheids with periods of less than ten days are infrequent. Five of the eight of shortest period are bright or highly reddened Cepheids, not near the galactic nucleus but in the neighborhood of the Sun.

An examination of the distribution of the periods, magnitudes and longitudes of all the 340 known galactic Cepheids confirms the conclusions based on the study of the five nuclear fields.

The correlation of period length with concentration of stellar mass is being further investigated through additional studies of the classical Cepheids in the two Magellanic Clouds.

* Cluster type Cepheids are peculiarly cosmopolitan in that they are the only variables known in the very low-density regions of the "galactic haze," far above and below the galactic plane; and they are also abundantly found not far from the centers of globular star clusters as well as in the galactic nucleus.

† The Cepheid X Sagittarii, the first variable in table 1, appears to be the "central" naked-eye star, bearing much the same relation to the galactic center as Polaris bears to the north pole; but d Ophiuchi, magnitude 4.37 and spectrum F5, is nearly as close to the center.

[1] These PROCEEDINGS, **26**, 105–115 (1940); *Harvard Reprint* 192.
[2] For example, *Harvard Reprint* 105 (1934).
[3] Shapley and Swope, *Harv. Ann.*, **90**, No. 5 (1934).
[4] See figure 3 of *Harvard Reprint* 58 (1929).

THE ABSORPTION OF BICARBONATE ION BY BARLEY PLANTS AS INDICATED BY STUDIES WITH RADIOACTIVE CARBON

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Communicated October 22, 1940

With the aid of radioactive carbon (C^{11}) Ruben and Kamen[1] have found that many heterotrophic systems reduce[2] appreciable quantities of C^*O_2 during respiration. Among the non-photosynthetic organisms investigated were yeast (baker's), *B. coli*, ground barely roots and ground liver (rat) tissue. The experiments involved the shaking of a distilled water suspension of the cells with a C^*O_2-air ($\sim$5% C^*O_2) mixture for periods varying from 5 to 100 minutes. The organisms were then freed of +4 carbon (i.e., CO_2, HCO_3^-, etc.) by boiling vigorously with strong acid. In all cases the tissues were found to retain radioactive carbon. This indicated the presence of radiocarbon in oxidation states lower than +4.

The significance of this "assimilation" of C^*O_2 is still uncertain. It is

possible that the apparent assimilation of C^*O_2 is merely an interchange of CO_2. If this is the explanation, then the reactions producing CO_2 in the respiratory process must be reversible. However, it is also possible that CO_2 is being used in a synthesis of compounds important in cellular metabolism. Ruben and Kamen favor the latter proposal even though it seems contrary to existing ideas on the metabolism of heterotrophic cells. Experiments are in progress to decide whether the C^*O_2 reduction is due to exchange or synthetic reactions.

The evidence pertaining to the assimilation of CO_2 by barley roots in water suspension would seem to have a direct bearing on the study of electrolyte accumulation by root cells. It admits of the possibility of a considerable absorption of HCO_3^- ions by the non-photosynthetic root cells through a process which results in the reduction and storing up of the carbon in the form of new compounds. Although many investigations indicate that H^+ and HCO_3^- ions may be intimately involved in the absorption process, the observations of the past have not been such as to suggest *the idea of absorption and assimilation of H_2CO_3 or HCO_3^- by tissues which at the same time excrete relatively large quantities of carbonic acid.*

The mechanism whereby certain favored cations and anions are absorbed by actively metabolizing root cells and held in the vacuoles at concentrations far exceeding those in the bathing culture medium is still unelucidated. Many observations point to the fact that the seat of the accumulation reaction must lie in the cell protoplasm. Although ions must pass through the fibrillated cellulose layer comprising the cell wall and presumably through a plasma membrane before encountering the protoplasm, it does not seem possible that the net result of the accumulation process will be completely explained on the basis of simple diffusion and membrane equilibria.

An important approach to the study of the accumulation process would be the determination of the relative proportions in which oppositely charged ions are absorbed by the roots. Heretofore difficulty has existed in evaluating the rôles of H^+ and HCO_3^- ions which are continuously excreted by the root as the result of the respiratory process. The possibility that HCO_3^- ion may play a part in the accumulation is suggested by the researches of Ruben and Kamen. These experiments, in addition to providing justification for further study regarding the nature of the "reduction" of CO_2 by plant roots, also suggest a means (i.e., the use of radioactively labeled HCO_3^- ion) for deciding the question whether or not individual ions can be absorbed by the plant through a process of ion exchange between the root cells and the culture medium. The experimental background leading to this question is contained in the considerable store of information which has accrued as a result of studies of actively absorbing plant cells in artificially prepared nutrient solutions. Only the more pertinent part of this

information will be mentioned here. For a more complete survey of these data the reader is referred to a recent review by Hoagland.[3]

A salient characteristic of the absorbing root system is the *apparently* independent entry of cations and anions from the culture solution into the root cells. For example, with K_2SO_4 culture solutions, the accumulation of K^+ by barley roots is much greater than the accumulation of $SO_4 = .$[3] Likewise, in the case of $Ca(NO_3)_2$ solutions, the accumulation of NO_3^- far exceeds the accumulation of Ca^{++}.[3] Moreover it has been observed that, for short intervals of time, the absorption of K^+ from $KHCO_3$ solutions is nearly equal to K^+ absorption from KCl solutions, although in the one case no accumulation HCO_3^- in the roots can be demonstrated, and in the other case the absorption of Cl^- is often nearly equivalent to the absorption of K^+.[4]

Of the numerous mechanisms which have been proposed, probably the one which most satisfactorily accounts for the observed alterations in the bathing culture medium during active electrolyte absorption by plant roots is the exchange absorption theory of Brooks[5] (compare also Lundegårdh[6]). According to this theory the initial step in the absorption of either a cation or an anion involves an exchange of the ion of the culture solution for an ion of like charge which is held on the surface of or within the protoplasm. The exchange hypothesis is in harmony with the experimental observation that an excess accumulation of anions over cations is usually compensated for in the culture solution by HCO_3^- ions, and an excess accumulation of cations over anions is compensated for by H^+ and other cations such as Na^+, Ca^{++} and Mg^{++} from the plant.[4] It may be noted, however, that the investigations of Hoagland and Broyer[4] and others do not support the view that a simple H^+ ion gradient mechanism between sap and external solution is involved.

Notwithstanding the fact that an exchange mechanism is in harmony with a number of observations, certain other observations are not easily explained. For example, in the case of barley roots, a certain *dependency* of cation absorption on anion absorption has been noted. Barley plants accumulate potassium much more readily from KCl solutions than from K_2SO_4 solutions of comparable concentrations.[4] Moreover, in the case of tissue such as potato tuber, an approximately equivalent absorption of cations and anions is observed under a variety of conditions for salts such as KCl and KBr.[7]

An essential consequence of the exchange theory is the independency of cation and anion assimilation, at least within certain limits. Obviously this fact can never be demonstrated until the rôles of H^+ and HCO_3^- ions, which are always present in actively absorbing root systems, have been evaluated. Until this has been accomplished experimentally, it can never be determined whether ions may enter the root cells independently through

exchange or must always enter as ion pairs of opposite sign. For instance, an apparently independent absorption of K^+ from a K_2SO_4 solution may actually be a simultaneous absorption of K^+ and HCO_3^- ions in exactly equivalent amounts; likewise an apparently independent absorption of NO_3^- from a $Ca(NO_3)_2$ solution in reality may be an equivalent uptake of H^+ and NO_3^- ions. In the following experiments with radioactively labeled HCO_3^-, some of the hitherto encountered uncertainty regarding the behavior of that ion has been eliminated. The experiments were designed primarily to decide whether in the case of a $KHCO_3$ culture solution, bicarbonate ion is reduced in amounts equivalent to the absorption of potassium ion.

Experiment I.—A preliminary absorption experiment was carried out on May 28. Young barley plants (approximately 3 weeks old) which had been grown by the technique of Hoagland and Broyer[8] were selected. The roots were so-called "low salt" roots, having an approximate K content of 40 milliequivalents per 100 g. of oven-dry material. The experiment was performed in the following manner. Twenty-one plants were decapitated about $^3/_4$ inch above the root-stem plate and fixed by means of non-absorbent cotton in holes in the wooden cover of the culture solution vessel. The roots were thoroughly washed in distilled water and then immersed in the KHC^*O_3 culture solution. The culture solution used was prepared as follows: 0.040 g. Merck's Reagent $KHCO_3$ was dissolved in 15 ml. distilled water. This solution was shaken with 2 cc. C^*O_2 (S.T.P.) in an evacuated bulb for 30 minutes at room temperature. The solution was then removed from the bulb and adjusted to 400 ml. with distilled water. The resulting culture medium was therefore very nearly 0.001 *N* in HCO_3^- ion. The volume of solution used in the experiment was 349 ml.

The system was aerated with a moderate stream of air throughout the period of absorption by the roots which was of exactly 64 minutes' duration. During the absorption all solution which exuded from the cut ends of the decapitated plants was collected by means of an eye-dropper and placed in a weighed weighing bottle containing 1 ml. of 0.1 *N* NaOH. At the end of the absorption period, the plants were removed from the culture medium and thoroughly washed with tap and distilled water. The plants were then cut into small pieces and placed in a distillation flask containing about 150 ml. distilled water. The $CO_2(HCO_3^-, CO_3^=$, etc.) was liberated from the suspension by boiling with an excess of H_2SO_4. The distillate was collected in a small volume (15 ml.) of NaOH solution. Following the distillation the volume of the NaOH solution was determined accurately. This solution contained all of the carbon present in the plants as +4 carbon. A measured aliquot of the solution was counted, and the total count of C^* present in the plant as HCO_3^- or H_2CO_3 calculated. The roots were then removed from the distillation flask and boiled for about 10 minutes in 150

ml. of 95% ethyl alcohol. The alcohol extract was combined with the acid solution in the distillation flask. The resulting solution was filtered and adjusted to volume. A measured aliquot of this solution was counted and the total count for the solution calculated. This count corresponded to the C* present in the plants as reduced carbon.[9] Finally the total count of the NaOH solution in the weighing bottle which contained the exudate was made. This count corresponded to the C* present in the exudate both as +4 carbon and as reduced carbon. The estimated volume of the exudate was 0.05 ml.

The distribution of radioactive carbon in the plant at the conclusion of the absorption period is given in the following table. The amounts of C* are expressed in counts per minute.

Total C* (mostly HCO_3^-) originally available to plants from culture	= 1.18×10^6	counts/min.
Total C* found in plants as HCO_3^- or H_2CO_3	= 1.60×10^3	"
Total C* found in plants as reduced carbon	= 4.56×10^4	"
Total C* found in exudate	= 12.7	"

Of the radioactive carbon originally available, the following percentages were retained by the plants:

Percentage retained as HCO_3^- or H_2CO_3	= 0.14%
Percentage retained as reduced carbon	= 3.86%
Percentage in exudate	= 0.0011%
Total percentage retained by plants	4.00%

Experiment II.—A much more complete absorption experiment was carried out on June 11. The plants used had been germinated May 14 and planted May 20. The K content of the roots of the 21 plants used in the experiment was 0.603 milliequivalent K. On the dry basis this corresponded to about 41 millequivalents per 100 g. The experiment was carried out in the same manner as I with two exceptions. The absorption period was 92 minutes and the volume of culture solution used (0.001 *N* $KHCO_3$) was 361 ml. The following additional measurements were made:

(*a*) The volume of the exudate was measured (by weighing).

(*b*) The pH and total CO_2 content of the culture solution at the beginning and at the end of the absorption period were determined.

(*c*) The loss of potassium from the culture solution during the absorption period was measured.

The results of measurements made on the culture solution before and after the 92-minute absorption period, together with the results of measurements on the plants and exudate, are presented in the following table:

Initial pH of culture solution	=7.81 (23°C.)
Final pH of culture solution	= 7.43 (23°C.)
Initial CO_2 content of culture solution	= 0.362 millimol
Final CO_2 content of culture solution	= 0.209 millimol
Loss of CO_2 from culture solution (chiefly as HCO_3^-)	= 0.153 millimol
Total potassium absorbed by plants	= 0.142 milliequivalent
Initial activity of culture solution (total)	= 4.16×10^6 counts/min.
Final activity of culture solution	= 1.15×10^6 " "
Activity in plants represented by +4 carbon	= 0.091×10^6 " "
Activity in plants represented by reduced carbon	= 2.050×10^6 " "
Total activity in exudate	= 0.007×10^6 " "
Total labeled carbon which entered plants	= 2.148×10^6 " "
Volume of culture solution	= 361 ml.
Volume of exudate	= 0.353 ml.

From the data it is evident that, of the radiocarbon originally available in the culture solution, the following percentages were retained by the plants:

Percentage retained as HCO_3^- or H_2CO_3	= 0.22%
Percentage retained as reduced carbon	= 4.93%
Percentage in exudate	= 0.02%
Total percentage retained by plants	= 5.1%

Discussion of Results and Conclusions.—In view of the differences in absorption period and volume of exudate, the results of Experiments I and II are in essential agreement. Radioactive carbon, present in the culture solution largely as $HC^*O_3^-$ ion, was definitely assimilated by the plant roots. Under the conditions of the experiments, 4–5 per cent of the originally available radiocarbon was retained by the tissue. Chemical examination of the tissue revealed that the absorbed C* was chiefly present in the plants as reduced carbon. Only about 4 per cent of the radioactive carbon present was H_2CO_3 or HCO_3^-. A minute, although significant fraction of the absorbed C* was exuded at the cut ends of the plants (0.3% in Experiment II).

The radioactivity in the exudate due to reduced carbon and that due to +4 carbon (HCO_3^-) may be estimated as follows:

From previous experiments with barley roots it is known that concentrations of ions in the exudate rarely if ever exceed those in the sap expressed from the roots. Since the approximate volume of the sap in Experiment II was 14.9 ml., the concentration of labeled +4 carbon in the sap, expressed in counts per minute per ml., was 6×10^2 counts/min./ml. From the data given above the total concentration of C* in the exudate was equivalent to 20×10^2 counts/min./ml. This indicates that a minimum of

70% of the radioactive carbon in the exudate was present in the reduced form.

Experiments comparable with Experiment II allow for a maximum of 0.2 millimol of CO_2 being respired in the 92-minute absorption period. Owing to the exchange of CO_2 between $HC^*O_3^-$ of the culture medium and inactive CO_2 excreted by the plants and CO_2 of the air which was used for aeration, the isotopic mol fraction of C* changed continuously throughout the absorption period. Thus in Experiment II the specific activity of the culture solution at the start of the absorption period was 11.49×10^6 counts per minute per millimol CO_2 and 5.52×10^6 counts per minute per millimol at the end.

The calculation of the amount of bicarbonate ion assimilated by the roots depends on the estimation of the average specific activity of $HC^*O_3^-$ at the outer surface of the protoplasm. On the assumption that the average specific activity at the protoplasmic surface was the same as the average specific activity calculated for the culture solution as a whole, the activity of the roots in Experiment II was equivalent to an assimilation of 0.025 millimol of HCO_3^-. However, the possibility of a lag in equilibrium between the region near the surface of the protoplasm and the rest of the culture solution is not inconceivable. Thus, this region may have been characterized by a lower specific activity than that calculated for the entire culture. This possibility would allow for the assimilation of a maximum of 0.06 millimol of HCO_3^- in Experiment II. When these values are compared with the value for the potassium absorption (0.142 millimol), it becomes evident that a several-fold absorption of K^+ over HCO_3^- must have taken place.

This result speaks for the independent retention of cations by plant roots, a fact which lends substantial support to the exchange absorption theory. It should be emphasized, however, that this conclusion applies to the net result of the accumulation reaction which presumably takes place in the cell protoplasm. The results of these experiments in no way preclude the possibility that K^+ and HCO_3^- may penetrate the root cell as far as the protoplasmic surface in chemically equivalent amounts.

Further, it should be recalled[4] that when K^+ is accumulated from a $KHCO_3$ solution, organic acids are synthesized and enter into equilibrium with the accumulated K^+ ions. K^+ and HCO_3 ions might enter the protoplasm and after reaction with organic acid most of the CO_2 be eliminated.

The total CO_2 determinations on the culture solution of Experiment II show that 0.153 millimol was lost from the bathing culture medium. In view of the pH change and the fact that this quantity of CO_2 could not have been reduced by the roots, one mechanism which would account for the loss would be an equivalent exchange of K^+ ions of the culture solution

for H^+ ions in the roots followed by the equivalent removal of H^+ and HCO_3^- (as CO_2) from the system through aeration.

Acknowledgments.—The authors wish to express their appreciation to Professor D. R. Hoagland for his helpful suggestions. We are indebted to Dr. M. D. Kamen and Professor E. O. Lawrence for the radioactive carbon.

Clerical assistance in the preparation of this publication was furnished by the personnel of the Work Projects Administration Official Project No. 65-1-08-91-B-10.

[1] Ruben, S., and Kamen, M. D., *Proc. Nat. Acad. Sci.*, **26**, 418–422 (1940).

[2] The terms oxidation and reduction usually imply a loss or gain of electrons. This idea has no theoretical significance when applied to the carbon compounds because the electron pairs are shared. While this limitation is generally recognized, these terms are widely used for the sake of convenience and brevity. We shall employ the usual terminology (cf. "The Oxidation States of the Elements and Their Potentials in Aqueous Solutions," W. M. Latimer, p. 118, Prentice-Hall, New York, 1938) which does not involve energy relationships or bond types, but refers to the kind of atom to which carbon is bonded.

[3] Hoagland, D. R., *Cold Spring Harbor Symposia Quant. Biol.* (in press), 1940.

[4] Hoagland, D. R., and Broyer, T. C., *Amer. Jour. Bot.*, **27**, 173–185 (1940).

[5] Brooks, S. C., *Trans. Faraday Soc.*, **33**, 1002–1006 (1937).

[6] Lundegårdh, H., *Die Naturwissenschaften*, **20**, 313–318 (1935).

[7] Steward, F. C., and Harrison, J. A., *Ann. Bot. N. S.*, **3**, 427–454 (1939).

[8] Hoagland, D. R., and Broyer, T. C., *Plant Physiol.*, **11**, 471–507 (1936).

[9] The efficiency of the isolation of +4 carbon and reduced carbon was verified by separate experiments. The boiling with alcohol resulted in the complete removal of active carbon from the solid part of the suspension.

A REMARK ON GENERAL FUCHSIAN GROUPS

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Communicated November 7, 1940

In the theory of the curves of genus 1 and the related elliptic functions a fundamental rôle is played by the modular group and its invariant functions (*modulfunctions*). Siegel generalized this group and these functions to the most general algebraic curves in two papers in *Annals of Mathematics*, **36**, 527–606 (1935), and in *Mathem. Annalen*, Bd. 116, pp. 617–657. In another paper in *Annals of Mathematics*, **41**, 488–494 (1940), Sugawara generalizes some of Siegel's results to a class of groups, which are the generalization of Poincaré's Fuchsian groups, and which the author calls "*general Fuchsian groups.*" In this paper the author finds also a geometry, which, in his case, corresponds to Lobatschevski's geometry, so

useful in Poincaré's theory. We can arrive at Sugawara's geometrical theorems by means of elementary and general methods[1] and theorems.

Let A, B, C, D, X, Y be real matrices of order n; we suppose that the determinant of Y is not equal to zero, and will write

$$Z = X + iY, \bar{Z} = X - iY.$$

We shall define a new matrix $Z_1 = X_1 + iY_1$ (and $\bar{Z}_1 = X_1 - iY_1$), by supposing

$$Z_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.Z. \tag{1}$$

By (1) we mean that

$$Z_1 = (AZ + B)(CZ + D)^{-1} \text{ or } Z_1(CZ + D) = AZ + B. \tag{2}$$

If P, Q, R, S are matrices of order n, and if

$$Z_2 = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.Z_1 = (PZ_1 + Q)(RZ_1 + S)^{-1}$$

we easily find that

$$Z_2 = \begin{pmatrix} H & K \\ L & M \end{pmatrix} Z = (HZ + K)(LZ + M)^{-1},$$

in which

$$\begin{pmatrix} H & K \\ L & M \end{pmatrix} = \begin{pmatrix} PA + QC & PB + QD \\ RA + SC & RB + SD \end{pmatrix}$$

is the product $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the preceding matrices (of order $2n$). Therefore:

If a set of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *of order 2n forms a group, the corresponding transformations (1) or (2) also form a group.*

By differentiating (2) we get (by considering A, B, C, D as given and constant)

$$(A - Z_1C)dZ = (dZ_1)(CZ + D). \tag{3}$$

By considering the imaginary parts of the members of (2) we get

$$Y_1(C\bar{Z} + D) = (A - Z_1C)Y \text{ or } Y^{-1}(A - Z_1C)^{-1} = (C\bar{Z} + D)^{-1}Y_1^{-1} \tag{4}$$

By multiplying (3) and (4) we find that

$$Y^{-1}dZ = (C\bar{Z} + D)^{-1}Y_1^{-1}(dZ_1)(CZ + D) \tag{5}$$

and therefore also (since A, B, C, D, Y, Y_1 are real)

$$Y^{-1}d\bar{Z} = (CZ + D)^{-1}Y_1^{-1}(d\bar{Z}_1)(C\bar{Z} + D). \tag{6}$$

By multiplying (5) and (6) we get finally:

$$Y^{-1}dZY^{-1}d\bar{Z} = (C\bar{Z} + D)^{-1}Y_1^{-1}(dZ_1)Y_1^{-1}(d\bar{Z}_1)(C\bar{Z} + D).$$

Therefore the two matrices

$$Y^{-1}(dZ)Y^{-1}(d\bar{Z}) \text{ and } Y_1^{-1}(dZ_1)Y_1^{-1}(d\bar{Z}_1) \tag{7}$$

are similar to each other and are also similar to

$$dZ(Y^{-1})(d\bar{Z})Y^{-1} = Y[Y^{-1}(dZ)Y^{-1}d\bar{Z}]Y^{-1} \text{ and } (dZ_1)Y_1^{-1}(d\bar{Z}_1)Y_1^{-1}. \tag{8}$$

Therefore

$$ds^2 = \text{Trace of } Y^{-1}(dZ)Y^{-1}(d\bar{Z}) = \text{Trace of } (dZ)Y^{-1}(d\bar{Z})Y^{-1} \tag{9}$$

is invariant under every transformation (1); and *every transformation (1) is a movement in the Riemannian geometry defined by the linear element (9).*

We have found this theorem only under the hypothesis that Y^{-1} exists (the determinant of Y is not equal to zero) and that (1) exists (the determinant of $CZ + D$ is not equal to zero).

This linear element (9) is real. If $Y^{-1}dZ = (u_{rs})$, the element (9) is equal to $\sum u_{rs}\bar{u}_{sr}$. By interchanging r with s, we prove that it is equal to the conjugate element and that consequently it is real. If $Z = (x_{rs} + iy_{rs})$ and $Y^{-1} = y'_{rs}$ (in which y'_{rs} is the algebraic complement of y_{sr}), an easy calculation proves that:

$$ds^2 = \sum y'_{rj}y'_{hi}(dx_{jh}dx_{ir} + dy_{jh}dy_{ir}) \tag{10}$$

I have demonstrated[2] that a group of movements in a Riemannian real *definite* geometry, which contains no infinitesimal transformations, is properly discontinuous.

Consequently: If a group G of transformations (1) or (2) transforms into itself a region R, in which our element (10) is definite, and if it contains no infinitesimal transformations, it is properly discontinuous in R.

For instance, let Γ be the group of the transformations (1) which satisfy Siegel's equations

$$A'C = C'A;\ B'D = D'B;\ A'D - C'B = E = \text{identity}$$

or the equivalent equations

$$AB' = BA';\ CD' = DC';\ AD' - BC' = E.$$

This group transforms into itself the region R of the *symmetrical matrices* $Z = X + iY$, *for which* $Y > 0$ (or the quadratic form corresponding to Y is definite positive). And in this region it is transitive; for every point Z_1

of R, we can find a transformation of Γ which carries Z_1 into $Z = iE$. Since the linear element is invariant, and Y is the identity E, the linear element in the neighborhood of Z_1 in R becomes equal to the trace of $dZd\bar{Z}$ which is

$$\sum dz_{rs}d\bar{z}_{sr} = \sum(dx^2_{rs} + dy^2_{rs}) \quad \text{(because } z_{sr} = z_{rs}\text{)}$$

and this form is obviously definite. We can also demonstrate, almost without calculations, that Γ is in R a group of movements for a real definite Riemannian geometry. Γ is transitive in R; and the transformations of Γ which carry $Z = iE$ into itself are obviously the transformations

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ (if } A'B = B'A \text{ and } A'A + B'B = E\text{)}.$$

This transformation carries the point $iE + dZ$ into a point $iE + dZ_1$, and obviously

$$dZ = (A + iB)^{-1}(dZ_1)(A - iB).$$

Therefore the trace of $dZ\,d\bar{Z}$ is invariant; since dZ is symmetric, this linear element is real and definite; and by means of a general theorem[3] we can deduce that Γ is in R a group of movements in a real definite Riemannian geometry.

The subgroups of Γ without infinitesimal transformations are called *general Fuchsian* groups, because they are the generalization of Poincaré's Fuchsian groups. We find consequently Sugawara's theorem:

Every Fuchsian group is properly discontinuous in R.

The most important Fuchsian group of order n is the modular group studied by Siegel, because it is the most important generalization of the classical modular group.

For $n = 2$ I could demonstrate that Γ is the group of the real conformal transformations of the euclidean indefinite geometry defined by supposing $ds^2 = dz_{11}dz_{22} - dz^2_{12}$.[4] Therefore: *The group of the conformal real transformations in a euclidean indefinite geometry of three dimensions can be considered as a group of movements in a real definite Riemannian geometry of six dimensions.*

But for the modular group G Siegel went much farther. He determined its fundamental region by making use of Minkowski's results concerning the reduction of quadratic forms. It might be very interesting if one could succeed in finding Siegel's and Minkowski's theorems by making use of the general methods, by means of which one can very often find the fundamental region of a group of movements in a real definite Riemannian geometry.

Some Remarks.—I. If $P = (p_{rs})$ is a symmetrical matrix, and Q another matrix, if

$$P_1 = Q'PQ,$$

let us consider the elements of P_1 as functions of the elements p of P (while Q is constant); it is obvious that the absolute value of the Jacobian of these functions is equal to the absolute value of

$$|\det Q|^{-(n+1)} \ (n = \text{order of } P, Q, P_1).$$

For the transformations of Γ the factor $A - Z_1C$ which we find in (3) and (4) is equal to $(ZC' + D')^{-1}$, so that we can also write for the points of the corresponding region R:

$$dZ_1 = (ZC' + D')^{-1}(dZ)(CZ + D).$$

Let us consider the elements of Z_1 as functions of the elements of Z. The absolute value of their Jacobian will be equal to

$$\Delta^{-(n+1)} \text{ [in which } \Delta = \text{determ}(CZ + D) = \text{determ}(ZC' + D')].$$

This Jacobian plays a fundamental rôle in Eisenhart's generalized series. In an analogous way we can find the discriminant of the preceding linear element.

II. Prof. Siegel has remarked that the geodesics of the definite geometry related to the group Γ are given by the following equation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .Z = i \begin{pmatrix} p_1^t & 0 \ldots 0 \\ 0 & p_2^t \ldots 0 \\ \ldots\ldots\ldots \\ 0 & 0 \ldots p_n^t \end{pmatrix}$$

Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is any matrix of Γ, t is a real variable, $p_1, p_2, \ldots, p_n$ are positive numbers $\neq 1$. This can be proved by an application of Cartan's idea for constructing the geodesics in a symmetrical space (Siegel).

III. By means of the transformation

$$W = (Z - iE)(Z + iE)^{-1} \quad (E = \text{identical matrix})$$

Sugawara transformed the preceding region, related to the group Γ, into a bounded region, and could therefore make use of Poincaré's methods in order to prove the convergence of some series.

IV. The group Γ may be *extended* by means of the *reflection* U defined by

$$Z_1 = -\bar{Z}.$$

This is obvious, because $U^2 = E$ and, if $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a transformation of Γ,

$$U^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}U = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

which is another transformation of Γ; consequently the groups Γ and $U^{-1}\Gamma U = U\Gamma U$ are identical.

[1] In the analytical part of his paper, Sugawara generalizes also Siegel's series. But it is not yet proved that Sugawara's functions are not identically equal to zero.

[2] Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe (Bologna, Zanichelli), page 126.

[3] Sugli spazii che ammettono ungruppo continuo di movimenti; *Annali di Matematica*, tomo 8, serie 3, 39–81 (1902).

[4] Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe (Bologna, Zanichelli), page 399.

THE DISTANCE IN GENERAL FUCHSIAN GEOMETRIES

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Communicated November 7, 1940

In a preceding note printed in these PROCEEDINGS (p. 695, this issue) I occupied myself with the most general Fuchsian geometries; their geodetic lines were determined by Prof. Siegel. By using Siegel's theorem we can deduce the geodetical distance between two matrices $Z = X + iY$, $Z_1 = X_1 + iY_1$, in which X, X_1, Y, Y_1 are symmetrical matrices of order n and Y, Y_1 are, moreover, definite positive. By E, N we indicate the identical and the null matrix. If P is a positive diagonal matrix of order n, its diagonal elements $p_1, p_2, \ldots, p_n$ are positive, and, according to Siegel's theorem, the point iP^t generates a geodetic line (t is a real parameter, P^t is the diagonal matrix, the elements of which are $p_1^t, p_2^t, \ldots, p_n^t$; we shall often write $\sqrt{P}$ instead of $P^{1/2}$). The linear element of the Fuchsian geometry is

$$ds^2 = \text{trace } \frac{1}{Y}\,dZ\,\frac{1}{Y}\,d\bar{Z} \quad (\bar{Z} = X - iY). \tag{1}$$

Therefore the distance between the points $t = t_1$, $t = t_2$ of the preceding geodetic line is

$$\sqrt{(\log p_1)^2 + (\log p_2)^2 + \ldots + (\log p_n)^2}\;|\,t_2 - t_1\,|\,. \tag{2}$$

If $Z = X + iY$ is one of our matrices, the (non-modular) transformation $\begin{pmatrix} 0 & -Y \\ E & -X \end{pmatrix}$ carries Z into iE,[1] and $\begin{pmatrix} -M & N \\ -N & -M \end{pmatrix}$ is the most general linear transformation, which carries iE into itself (M, N matrices of order n). Therefore

$$\begin{pmatrix} -M & N \\ -N & -M \end{pmatrix} \begin{pmatrix} O & -Y \\ E & -X \end{pmatrix} = \begin{pmatrix} N & MY - NX \\ -M & NY + MX \end{pmatrix}$$

is the most general linear transformation which carries Z into iE. It is modular if and only if

$$N'M = M'N; \quad Y(M'M + N'N)X = X(M'M + N'N)Y; \; E = (M'M + N'N)Y.$$

The second condition is a consequence of the third, and consequently can be disregarded. In order to simplify the third condition, let us choose an arbitrary non-singular matrix Γ, and put

$$N = C\Gamma^{-1}, \; M = A\Gamma^{-1}.$$

The first condition becomes

$$C'A = A'C. \tag{3}$$

The third condition $E = (\Gamma')^{-1}(A'A + C'C)\Gamma^{-1}Y$ is turned into the simpler condition

$$A'A + C'C = E \tag{4}$$

if we choose Γ in such a way that

$$Y = \Gamma\Gamma' \tag{5}$$

and this is obviously possible, because Y is positive definite. If $X = 0$, $Y = P$, (P diagonal), we can suppose $\Gamma = \Gamma' = \sqrt{P}$. Therefore $\begin{pmatrix} \frac{1}{\sqrt{P}} & \sqrt{P} \\ 0 & \end{pmatrix}$ carries iP into iE, and consequently $\begin{pmatrix} 0 & -\sqrt{P} \\ \frac{1}{\sqrt{P}} & 0 \end{pmatrix}$ carries iE into iP. Therefore

$$T = \begin{pmatrix} 0 & -\sqrt{P} \\ \frac{1}{\sqrt{P}} & 0 \end{pmatrix} \begin{pmatrix} N & MY - NX \\ -M & NY + MX \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{P}A\Gamma^{-1} & -\sqrt{P}(C\Gamma' + A\Gamma^{-1}X) \\ \frac{1}{\sqrt{P}}C\Gamma^{-1} & \frac{1}{\sqrt{P}}(A\Gamma' - C\Gamma^{-1}X) \end{pmatrix} \tag{6}$$

is the most general modular transformation, which carries Z into iP (under the conditions (3), (4), (5)).

If Z_1 is another of our matrices, and P_1 another positive diagonal matrix, and if A_1, C_1, Γ_1 satisfy the equations analogous to (3), (4), (5), we shall deduce the most general modular transformation T_1, which carries Z_1 into iP_1 and which is given by a formula quite analogous to (6). Can T and T_1 be identical? Or, in other words, can the same transformation carry Z into iP and Z_1 into iP_1? If we write that the coefficients of T_1 are equal to the corresponding coefficients of T, and if we write also the equations analogous to (3), (4), (5), we get:

$$C_1'A_1 = A_1'C_1 \tag{3.1}$$

$$A_1'A_1 + C_1'C_1 = E \tag{4.1}$$

$$Y_1 = \Gamma_1\Gamma_1' \tag{5.1}$$

$$A_1 = \sqrt{\frac{P}{P_1}}\,AK, \quad C_1 = \sqrt{\frac{P_1}{P}}\,CK \tag{7}$$

$$\sqrt{P}\,(C\Gamma' + A\Gamma^{-1}X) = \sqrt{P_1}\,(C_1\Gamma_1' + A_1\Gamma_1^{-1}X_1) \tag{8}$$

$$\frac{1}{\sqrt{P}}\,(A\Gamma' - C\Gamma^{-1}X) = \frac{1}{\sqrt{P_1}}\,(A_1\Gamma' - C_1\Gamma^{-1}X_1) \tag{9}$$

in which

$$K = \Gamma^{-1}\Gamma_1 \quad (\Gamma_1 = \Gamma K;\ \Gamma_1' = K'\Gamma'). \tag{10}$$

By virtue of (7) and (10), the (8), (9) are equivalent to:

$$C - QC\Delta = AH \tag{8.1}$$

$$-A + Q^{-1}A\Delta = CH, \tag{9.1}$$

in which

$$Q = P_1P^{-1} = P^{-1}P_1;\ \Delta = KK';\ H = \Gamma^{-1}(X_1 - X)(\Gamma')^{-1}. \tag{11}$$

(H is symmetrical, since X_1, X are symmetrical.) By means of (7) we realize that (3.1) is equivalent to (3), and that (4.1) is equivalent to

$$A'Q^{-1}A + C'QC = \Delta^{-1}. \tag{4.2}$$

If $P = E$, and $Q = P_1$ has the elements $q_1, q_2, \ldots, q_n$, the distance from Z to Z_1 (or from P to P_1) is

$$\text{distance} = \sqrt{(\log q_1)^2 + (\log q_2)^2 + \ldots + (\log q_n)^2}. \tag{12}$$

In order to calculate this distance, we must study *the system of the five equations* (3), (4), (4.2), (8.1), (9.1), *in which A, C, Q are the three unknown matrices*. We remark that $\Delta = KK' = \Gamma^{-1}\Gamma_1\Gamma_1'\Gamma'^{-1} = \Gamma^{-1}Y_1(\Gamma')^{-1}$ is,

like Y_1, *a positive symmetrical matrix*. From (8.1), (9.1) we deduce that every element q_r of Q (which is also a diagonal positive matrix) satisfies *the fundamental equation*

$$\begin{vmatrix} E - q\Delta & H \\ -H & E - \frac{1}{q}\Delta \end{vmatrix} = 0. \qquad \text{(I)}$$

The demonstration is easy. By developing the equations (8.1) and (9.1) we get:

$$c_{rs} - q_r \sum_j c_{rj}\delta_{js} = \sum a_{rj}h_{js}; \quad -\sum c_{rj}h_{js} = a_{rs} - \frac{1}{q_r}\sum a_{rj}\delta_{js} \qquad \text{(8-9.2)}$$

($\delta_{js} = \delta_{sj}$, $h_{sj} = h_{js}$, c_{rs}, a_{rs} are the elements of Δ, H, C, A).

For every value of r ($r = 1, 2, \ldots, n$) we obtain consequently a system of $2n$ homogeneous linear equations, in $2n$ unknowns $c_{r1}, c_{r2}, \ldots, c_{rn}, a_{r1}, a_{r2}, \ldots, a_{rn}$. The determinant of these equations (first member of I) must therefore be equal to zero.

By interchanging the rows or the columns, and by interchanging rows and columns, or by changing the sign of some rows, we easily realize that (I) is equivalent to:

$$\begin{vmatrix} E - \frac{1}{q}\Delta & H \\ -H & E - q\Delta \end{vmatrix} = 0, \text{ or } (\Gamma_1 2) \begin{vmatrix} E - q\Delta & H \\ H & -E + \frac{1}{q}\Delta \end{vmatrix} = 0 \text{ (I.1)}$$

From (I.1) we deduce:

The fundamental equation does not change if we write $\frac{1}{q}$ *in the place of the unknown* q (reciprocal equation).

From (I.2) we deduce another result. Let us consider $2n$ new variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Let $F(x)$ be the quadratic form of the x, the determinant of which is $E - q\Delta$, let $F_2(y)$ be the quadratic form of the y, the determinant of which is $-E + \frac{1}{q}\Delta$ and $H(x, y)$ the bilinear form of the x, y, the determinant of which is H: *a number* q *is a root of the fundamental equation if and only if the quadratic form*

$$F = F_1(x) + F_2(y) + 2H(x, y)$$

(in $2n$ variables x, y) *is singular*. (H is symmetrical, Δ is positive and symmetrical.) We shall prove: *Under these hypotheses the roots of* (*I*) *are real and positive* (and therefore (12) has a *real* meaning). We begin by remarking that (if Y and Y_1 are given) the equations (5) and (5.1) do not determine completely Γ and Γ_1. It is known that we can change Γ, Γ_1 in

such a way that $K = \Gamma^{-1}\Gamma$ and $\Delta = KK'$ become diagonal matrices (we know that Δ is also positive). Obviously, by changing Γ, Γ_1 we must change H also, according to (11). If δ_i are the (positive) elements of the (diagonal) matrix Δ, we have obviously

$$F = \sum m_i x_i^2 - \sum n_i y_i^2 + 2\sum h_{ij} x_i y_j$$

$$\left(m_i = 1 - q\delta_i;\ n_i = 1 - \frac{\delta_i}{q}\right)$$

1°. If F is singular, and q is real, it is not possible that all the coefficients m, n have the same sign. Let us suppose that the m, n are positive (if they were negative, we could study the form $-F$). We get

$$F = (x_1')^2 + (x_2')^2 + \ldots + (x_n')^2 - \varphi$$

in which

$$x_i' = \sqrt{m_i}\, x_i + \frac{1}{\sqrt{m_i}} \sum h_{ij} y_j$$

and

$$\varphi = \sum n_i y_i^2 + \sum_i \left(\frac{1}{\sqrt{m_i}} \sum_j h_{ij} y_j\right)^2.$$

The form φ does not depend upon the x'; if F is singular, φ is consequently singular also, and therefore may be equal to zero even if the (real) y are not all equal to zero. But this is not possible, because $n_i > 0$.

If a root q were negative, all the m, n would be positive; which is not possible; therefore the roots are positive; and the differences $\frac{1}{\delta} - q,\ q - \delta$ cannot have the same sign.

2°. The roots of (I) cannot be imaginary. If $q = \rho(\cos\theta + i \operatorname{sen}\theta)$ is an imaginary root ($\operatorname{sen}\theta \neq 0$), then

$$m_j = 1 - q\delta_j = 1 - r_j(\alpha + i\beta);\ n_i = 1 - \frac{\delta_i}{q} = 1 - s_j(\alpha - i\beta)$$

$$(r_j = \rho_j^{\delta};\ s_j = \frac{\delta_j}{\rho};\ \alpha = \cos\theta;\ \beta = \operatorname{sen}\theta \neq 0).$$

If F is singular, we can find $2k < 4n$ linear real forms u_α, v_α in x, y such that

$$F = \sum (u_\alpha + iv_\alpha)^2 \quad (\alpha = 1, 2, \ldots, k)(k < 2n).$$

By equating the imaginary parts we get

$$-\beta[\sum r_j x_j^2 + \sum s_j y_j^2] = 2\sum u_\alpha v_\alpha \ (j = 1, 2, \ldots, n;$$

$$\alpha = 1, 2, \ldots, k < 2n).$$

Since $k < 2n$, we can find real values of x, y, which are not all equal to zero, and which satisfy the k linear homogeneous equations $v_1 = v_2 = \ldots = v_k = 0$. Therefore the second member, and consequently also the first member of the last identity will be equal to zero; and this is not possible, because $\beta \neq 0$, $r > 0$, $s > 0$. All the roots $q_1, q_2, \ldots, q_n, q_{n+1}, \ldots, q_{2n}$ are consequently positive. We can suppose

$$q_1 \geq q_2 \geq \ldots \geq q_n \geq q_{n+1} \geq \ldots \geq q_{2n} \text{ or } q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots \leq q_{2n}.$$

From our theorems it follows that

$$q_{n+j} = \frac{1}{q_{n-j+1}} \quad (j = 1, 2, \ldots, n).$$

The $q_1, q_2, \ldots, q_n$, and therefore the distance (12), *are determined and real;* this distance is *a symmetrical but non-algebraic function of the* $q_1, q_2, \ldots, q_n$. We must now study the system of the equations (3), (4), (4.2), (8.1), (9.1).

For the sake of simplicity we study only the most general case, and suppose that $q_1, q_2, \ldots, q_n$ are different from each other; these roots will be simple $\left(\text{except } q_n \text{ if } q_n = 1 = \frac{1}{q_{n+1}}\right)$.[2]

Since Δ is diagonal, the equations (8.1), (9.1) or (8-9.2) become:

$$c_{rs}(1 - q_r\delta_s) = \sum_j a_{rj}h_{js}; \tag{8.3}$$

$$a_{rs}\left(-1 + \frac{1}{q_r}\delta_s\right) = \sum_j c_{rj}h_{js}. \tag{9.3}$$

If $c_{rs} = \gamma_{rs}$, $a_{rs} = \alpha_{rs}$ is a solution (γ_{rs}, α_{rs} not all equal to zero), also

$$c_{rs} = \rho_r\gamma_{rs}, \; a_{rs} = \rho_r\alpha_{rs} \; (\rho_r \neq 0 \text{ arbitrary}) \tag{13}$$

is a solution. We deduce at once

$$\sum_s c_{rs}a_{ls} - q_r \sum c_{rs}\delta_s a_{ls} = \zeta_{rl} \qquad \left(\zeta_{rl} = \sum_{js} a_{rj}h_{js}a_{ls}\right)$$

$$-\sum c_{rs}a_{ls} + \frac{1}{q_l}\sum c_{rs}\delta_s a_{ls} = \sum_{rl} \qquad \left(\sum_{rl} = \sum_{js} c_{rj}h_{js}c_{ls}\right) \tag{14}$$

in which ζ_{rl} and $\sum_{rl}$ are, like h_{rs}, symmetrical in r, t. Therefore

$$\left(\frac{1}{q_l} - q_r\right)c_{rs}a_{ls} = \frac{1}{q_l}\zeta_{rl} + q_r\sum_{rl},$$

or

$$\sum_s c_{rs}a_{ls} = Z_{rl} + \frac{q_r q_l}{1 - q_r q_l}(\zeta_{rl} + \sum_{rl}).$$

Therefore $\sum_s c_{rs}a_{ts}$ is symmetrical in r, t. (Our demonstration is not valid if $q_rq_t = 1$; but in this case $r = t = n$, $q_n = 1$ and our theorem is obvious.) Therefore

$$CA' = AC'. \tag{15}$$

From (14) we get also

$$\sum c_{rs}\delta_s a_{ts} = \frac{q_t}{1 - q_rq_t}(\zeta_{rt} + \sum_{rt}) \text{ (if } q_rq_t \neq 1). \tag{16}$$

From (8.3), (9.3) we deduce

$$\sum_s c_{rs}c_{ts} - q_r \sum c_{rs}\delta_s c_{ts} = \sum_{j,s} a_{rj}h_{js}c_{ts}$$

$$-\sum_s a_{rs}a_{ts} + \frac{1}{q_t}\sum_s a_{ts}\delta_s a_{rs} = \sum_{j,s} c_{tj}h_{js}a_{rs}$$

and, by subtracting,

$$\sum_s (c_{rs}c_{ts} + a_{rs}a_{ts}) = q_r \sum c_{rs}\delta_s c_{ts} + \frac{1}{q_t}\sum a_{ts}\delta_s a_{rs}. \tag{17}$$

By interchanging r with t and by subtracting, we get:

$$(q_r - q_t)\sum c_{rs}\delta_s c_{ts} + \left(\frac{1}{q_t} - \frac{1}{q_r}\right)\sum a_{ts}\delta_s a_{rs} = 0,$$

or, if $r \neq t$ and consequently $q_r \neq q_t$:

$$q_r \sum_s c_{rs}\delta_s c_{ts} + \frac{1}{q_t}\sum_s a_{ts}\delta_s a_{rs} = 0 \ (r \neq t). \tag{18}$$

From (17) and (18) we deduce that $CC' + AA'$ is a diagonal matrix; from (13) we get that the rth element of the diagonal is

$$\rho_r^2 \sum_s (\gamma_{rs}^2 + \alpha_{rs}^2)$$

and we can choose the ρ in such a manner that all these elements are equal to 1, or that

$$E = CC' + AA'. \tag{19}$$

From (15), (19) it follows that $\begin{pmatrix} A & -C \\ C & A \end{pmatrix}$ is a modular transformation [or that $(C + iA)(C' - iA') = E$]. Therefore also the inverse transformation $\begin{pmatrix} A' & C' \\ -C' & A' \end{pmatrix}$ is modular [or $(C' - iA')(C + iA) = E$], and consequently

$$C'A = A'C;\ E = C'C + A'A;$$

the equations (3) and (4) are satisfied. From (3), (8.1), (9.1) we get:

$$C'C - C'QC\Delta = C'AH = A'CH = -A'A + A'Q^{-1}A\Delta$$

or

$$C'C + A'A = (C'QC + A'Q^{-1}A)\Delta$$

or

$$E = (C'QC + A'Q^{-1}A)\Delta$$

which is identical with (4, 2). All our equations are satisfied, and our problem is solved; in order to find the distance between Z and Z_1 it is sufficient to make use of (12), if the q_i are the quoted roots of the fundamental equation (I). (It may be useful to consider $\left(q + \frac{1}{q}\right)$ as the unknown of this equation.)

Remarks.—I. From (17) and from (8.1), (9.1) we deduce

$$CC' + AA' = QC\Delta C' + Q^{-1}A\,\Delta A';\ CC' + AA' = QC\Delta C' - Q^{-1}A\,\Delta A' + AHC' - CHA'.$$

Consequently $AHC' = CHA'$, *and also* AHC' *is symmetrical;* if C is not singular, $C^{-1}AH$ is symmetrical too. (Don't forget that $C^{-1}A$ is symmetrical.)

II. If we do not make use of the matrices Γ, Γ_1, we can consider the transformations

$$\begin{pmatrix} \sqrt{P}\,A & -\sqrt{P}\,(CY + AX) \\ \frac{1}{\sqrt{P}}\,C & \frac{1}{\sqrt{P}}\,(AY - CX) \end{pmatrix}$$

which carry $X + iY$ into iP, and are modular if

$$A'C = C'A;\ A'A + C'C = Y^{-1}.$$

By using the preceding methods we find the equations

$$CY - QCY_1 = A(X_1 - X),\ -AY + Q^{-1}AY_1 = C(X_1 - X);$$

and the fundamental equation (I) becomes:

$$\begin{vmatrix} Y - qY_1 & X_1 - X \\ X_1 - X & -Y + \frac{1}{q}\,Y_1 \end{vmatrix} = 0 \qquad \text{(I, 3)}$$

(X_1, X symmetrical; Y, Y_1 symmetrical and positive definite). This last equation can be written immediately, if one has given the matrices $X + iY$,

$X_1 + iY_1$ and one wishes to calculate their distance by means of (12). It may perhaps be useful to consider $q + \frac{1}{q}$ as the unknown of this equation.

It is easy to verify this result for Lobacevsky's geometry ($n = 1$).

[1] By definition, if A, B, C, D are matrices of order n, the linear transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ carries Z into $(AZ + B)(CZ + D)^{-1}$.

[2] In order to study the particular cases when at least two of the q_i ($i \leq n$) are equal to each other, it is necessary to study the group of the modular transformations which carry into itself every point of the geodetic line joining the points iE, iQ. The study of this group is interesting and not difficult.

ABSTRACT DEFINITE BOUNDARY VALUE PROBLEMS

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Communicated November 12, 1940

In this note we describe the basis for an extensive further development of the abstract theory of boundary value problems initiated in a previous paper,[1] state some of the more important results of the extended theory and indicate some of its applications. A detailed discussion of the subject will appear elsewhere.

The present theory is aimed at those problems with which a semi-bounded symmetric form is associated—as, for example, the Dirichlet integral is associated with various problems involving Laplace's equation—and in consequence it is not as widely applicable as the theory developed in (A). However, there is great compensation for this in two important respects: first, where the present theory does apply, it provides at once a much more complete analysis than can be achieved on the basis of (A) alone; second, its basic concepts are much simpler than those of (A) and consequently the technical problems which arise in connection with its application to differential operators are considerably simpler.

We consider a bilinear non-negative definite symmetric form $G(f, g)$ defined for every pair of elements f, g in a dense linear subset $\mathfrak{G}$ of Hilbert space $\mathfrak{H}$ and require that G be closed in the following sense: if $\{f_n\}$ converges to f in $\mathfrak{H}$ and $\lim_{n\to\infty} G(f_n - f_m, f_n - f_m) = 0$, then f is in $\mathfrak{G}$ and

$$\lim_{n\to\infty} G(f_n - f, f_n - f) = 0.$$

In view of the fact that G is non-negative and closed, $\mathfrak{G}$ itself is a Hilbert space with inner product $G(f, g) + (f, g)$, and when we speak of $\mathfrak{G}$ as a space it is to be understood in this sense.

Before proceeding it is convenient to state an important result of Friedrichs[2] which plays a fundamental rôle in the theory under discussion.

FRIEDRICHS' THEOREM. *If G satisfies the requirements stated above, there exists one and only one self-adjoint transformation $S = S(G)$ in $\mathfrak{H}$ with $\mathfrak{D}(S) = \subseteq \mathfrak{G}$ such that $(f, Sg) = G(f, g)$ for all f in $\mathfrak{G}$ and all g in $\mathfrak{D}(S)$. If H is an arbitrary transformation with domain in $\mathfrak{G}$ such that $(f, Hg) = G(f, g)$ for all f in $\mathfrak{G}$ and all g in $\mathfrak{D}(H)$, then $H \subseteq S$.*

We now introduce our fundamental concept, an abstraction of the notion of boundary values, and state our basic theorem.

DEFINITION 1. *A closed linear transformation B from the Hilbert space $\mathfrak{G}$ to a unitary or Hilbert space $\mathfrak{N}$ is called a boundary operator on $\mathfrak{G}$ if the following conditions are satisfied:*

(1) *$\mathfrak{D}(B)$ is dense in the space $\mathfrak{G}$;*
(2) *the manifold of elements f of $\mathfrak{D}(B)$ such that $Bf = 0$ is dense in $\mathfrak{H}$;*
(3) *$\mathfrak{R}(B)$ is dense in $\mathfrak{N}$.*

The space $\mathfrak{N}$ is called the range space of B, and Bf is called the boundary value of f.

It is worth while to note that if G is bounded above, then $\mathfrak{G} = \mathfrak{H}$ and the only operator B satisfying the conditions of Definition 1 is identically zero. On the other hand if G is unbounded and $\mathfrak{N}$ is an arbitrary unitary or Hilbert space, one can show that there exists a boundary operator on $\mathfrak{G}$ with range-space $\mathfrak{N}$.

THEOREM 1. *Let B be a boundary operator on $\mathfrak{G}$. Then there exists one and only one pair of transformations T and N satisfying the following conditions:*

(1) $\mathfrak{D}(T) = \mathfrak{D}(N) \subseteq \mathfrak{D}(B)$;
(2) $\mathfrak{R}(T) \subseteq \mathfrak{H}$, $\mathfrak{R}(N) \subseteq \mathfrak{N}$;
(3) *the transformation L in $\mathfrak{H} \oplus \mathfrak{N}$ with domain the set of all vectors $\{f, Bf\}$, which takes $\{f, Bf\}$ into $\{Tf, Nf\}$ is self-adjoint;*
(4) $(f, Tg) + (Bf, Ng) = G(f, g)$ *for all f in $\mathfrak{D}(B)$ and all g in $\mathfrak{D}(T)$.*

To prove Theorem 1, one has only to note that, by virtue of the restrictions on B, G can be regarded as a form defined on the set of all $\{f, Bf\}$ in $\mathfrak{H} \oplus \mathfrak{N}$ and in this sense satisfied all the conditions of Friedrichs' theorem. Application of the latter then yields at once the result stated.

The significance of Definition 1 and Theorem 1 and the direction of the analysis based on them is now best revealed by means of some concrete examples.

EXAMPLE 1. The space $\mathfrak{H}$ is $\mathfrak{L}_2(a, b)$, $-\infty < a < b < \infty$; $G(f, g) = \int_a^b (pf'\bar{g}' + qf\bar{g})dx$, $p > 0$, $q \geqq 0$; $\mathfrak{N}$ is a two-dimensional unitary space and $Bf = \{f(a), f(b)\}$. In this case

$$Tf = -(pf')' + qf, \text{ and } Nf = \{-f'(a), f'(b)\}.$$

EXAMPLE 2. The space $\mathfrak{H}$ is $\mathfrak{L}_2(E)$, where E is the portion of the (x, y)-plane bounded by a smooth closed curve C;

$$G(f, g) = \int\int_E [p(f_x\bar{g}_x + f_y\bar{g}_y) + qf\bar{g}]dE,\ p > 0,\ q \geqq 0;$$

$\mathfrak{N}$ is the space $\mathfrak{L}_2(C)$ and Bf is the value of f on C. Here $Tf = -(pf_x)_x - (pf_y)_y + qf$, and $Nf = -p\partial f/\partial n$.

EXAMPLE 3. The space $\mathfrak{H}$ is the same as in Example 2;

$$G(f, g) = \int\int_E \nabla^2 f\, \nabla^2\bar{g}\, dE,$$

where ∇^2 is Laplace's operator; $\mathfrak{N}$ is the space $\mathfrak{L}_2(C) \oplus \mathfrak{L}_2(C)$ and $Bf = \{f(s), -\partial f/\partial n\}$. Here T is the biharmonic operator, $Tf = \partial^4 f/\partial x^4 + 2\partial^4 f/\partial x^2\partial y^2 + \partial^4 f/\partial y^4$, and $Nf = \{-\partial\nabla^2 f/\partial n, -\nabla^2 f(s)\}$.

In describing these examples we have for the sake of brevity passed over various function-theoretic details such as the definition of the set $\mathfrak{G}$, the precise conditions on the curve C in Examples 2 and 3, and the definition of boundary values and normal derivatives. These matters, however, will all be dealt with in subsequent papers on the applications of the present theory.

THEOREM 2. *The transformation T of Theorem 1 has domain dense in $\mathfrak{H}$ and thus T^* exists. The relation $T \supseteqq T^*$ holds and $\mathfrak{D}(T^*)$ consists of those and only those elements f of $\mathfrak{D}(T)$ such that $Bf = Nf = 0$. The operator T^* is Hermitian and non-negative definite.*

Two cases now arise according as the domain of T^* is or is not dense in $\mathfrak{H}$. In the former case T^* is symmetric, $\tilde{T}$ exists and is identical with T^{**}; in the latter, $\tilde{T}$ does not exist.

THEOREM 3. *Let $\mathfrak{A}$ be the manifold of zeros of B, $\mathfrak{M}$ the closure in $\mathfrak{H}$ of $\mathfrak{G} \ominus \mathfrak{A}$, and $S = S(G)$ the operator in $\mathfrak{H}$ associated with G by Friedrichs' theorem. Then $\mathfrak{D}(T^*)$ is dense in $\mathfrak{H}$ if and only if $\mathfrak{M} \cdot \mathfrak{D}(S) = \mathfrak{D}$.*

In the remainder of this note we assume that the condition of Theorem 3 is satisfied; this is the case in the examples given above and in all similar ones of which we have been able to conceive.

We now bring Theorem 1 into correlation with the theory of (A).

THEOREM 4. *Let A be the transformation with domain $B(T)$ which takes $\{f, Tf\}$ into $\{Bf, Nf\}$. Then A is a reduction operator for T in the sense of (A). The range-space of A is $\mathfrak{N} \oplus \mathfrak{N}$, and the unitary operator W in $\mathfrak{N} \oplus \mathfrak{N}$ associated with A takes $\{h, k\}$ into $\{k, -h\}$.*

Thus, when the condition of Theorem 3 is satisfied, the results of (A) apply to yield an analysis of the situation described in Theorems 1 and 2 with reference to the problem of characterizing by boundary conditions those self-adjoint transformations S such that $T^* \subseteqq S \subseteqq T$. We shall not, however, discuss this aspect of the theory in the present note; its nature is

adequately indicated in an earlier communication[3] dealing with differential operators of the type appearing in Example 2 above.

Rather we wish to emphasize here that portion of the theory which stems directly from Theorem 1 and which provides the machinery for the operational treatment of many important problems in differential equations which had no precise interpretation in terms of the concepts of (A) alone. Beside thorough examination of the various cases which arise under Definition 1 and Theorem 1, which is, of course, technically necessary, this theory comprises (1) a careful analysis of the abstract analogues of the Dirichlet and Neumann problems of potential theory, of the so-called mixed boundary value problem, and of what may be termed abstract boundary value problems of the third kind; (2) a detailed study of the homogeneous boundary conditions related to each of the above problems, and of the characteristic value problems involving them; (3) the development of an elementary theory of variations for forms G satisfying the conditions stated at the beginning of this note and certain additional requirements, with reference to those variational problems which are equivalent to important boundary and characteristic value problems.

Applied to ordinary differential forms and operators this theory yields at once a considerable portion of the classical theory of boundary and characteristic value problems; indeed, it is not yet determined how far one can proceed abstractly in this direction. Moreover, applied to partial differential forms and operators, the theory achieves a considerable generalization and refinement of known results in several directions; this assertion is true in particular with reference to Examples 2 and 3 above, and will be amplified in a subsequent paper on differential operators. Finally, the abstract theory clarifies to a great extent many familiar analogies between boundary value problems involving various kinds of differential operators and systems of differential operators; it now becomes possible to regard these similar problems no longer merely as analogous one to the other but rather as special instances of the same abstract problem.

To conclude, and to amplify and illustrate the preceding general remarks, we state precisely some of our results, leaving to the reader their interpretation in terms of the examples given above.

First, let us consider the boundary condition $Bf = 0$, which is of special importance in all of the examples we have given and in many others. To begin we recall a result of Friedrichs.[2] If H is a closed linear non-negative definite symmetric transformation in $\mathfrak{H}$, the form (f, Hg) has a unique closed linear extension G_0 in $\mathfrak{H}$ which is determined by completing the space $\mathfrak{D}(H)$ with inner product $(f, Hg) + (f, g)$. If $S(G_0)$ is the self-adjoint transformation associated with G_0 by Friedrichs' theorem above, then $S(G_0) \supseteq H$. We call $S(G_0)$ Friedrichs' extension of H.

THEOREM 5. *Let S be the contraction of $\hat{T}$ with domain the set of elements*

f of $\mathfrak{D}(T)\cdot\mathfrak{D}(B)$ *such that* $Bf = 0$. *Then S is Friedrichs' extension of* T^*.

THEOREM 6. *Let* $\mathfrak{A}$ *be the orthogonal complement in the space* $\mathfrak{G}$ *of the manifold of zeros of B. Then a necessary and sufficient condition that the relations* $S \subseteqq T$ *hold is that for some constant k and all f in* $\mathfrak{A}$, *we have*

$$(f, f) \leqq k^2(Bf, Bf).$$

Now let us turn to the problem of solving the system

$$\tilde{T}f = 0, Bf = h, \qquad (*)$$

where h is some element of $\mathfrak{N}$, noting that when $G(f, f)$ is the Dirichlet integral and Bf the ordinary boundary value of f (cf. Example 2 above), this is just the Dirichlet problem. In the projected, comprehensive discussion of this subject, this important problem is treated in detail; here we merely state one result.

THEOREM 7. *Let the manifold* $\mathfrak{M}$ *of singularities of G have a finite dimension number and belong to the domain of B. Let G be positive definite in* $\mathfrak{H} \ominus \mathfrak{M}$. *Then the system* (*) *has a solution f for every h in* $\mathfrak{R}(B)$. *This solution is unique if and only if the transformation S of Theorem 5 has an inverse.*

THEOREM 8. (*Dirichlet principle.*) *Let the hypothesis of Theorem 7 be satisfied. Let h be an arbitrary element of* $\mathfrak{R}(B)$ *and let* $\mathfrak{F}(h)$ *be the set of all f in* $\mathfrak{D}(B)$ *such that* $Bf = h$. *Then* $G(f, f)$ *has a minimum on* $\mathfrak{F}(h)$ *which it attains if and only if f is a solution of the system* (*).

Finally, let us consider the condition $Nf = 0$ and the abstract Neumann problem. Here again we restrict our attention to special results.

THEOREM 9. *Let B be bounded and let S be the contraction of T whose domain consists of those and only those elements f of* $\mathfrak{D}(T)$ *such that* $Nf = 0$. *Then S is the transformation S(G) associated with G by Friedrichs' theorem.*

THEOREM 10. *Let B be bounded and let the hypothesis of Theorem 7 be satisfied. Let* $\mathfrak{P}$ *be the manifold in* $\mathfrak{N}$ *of boundary values of elements of* $\mathfrak{D}$ (T) *such that* $Tf = 0$, $Nf = 0$. *Then the system*

$$Tf = 0, Nf = h$$

has a solution for every h in $\mathfrak{N} \ominus \mathfrak{P}$. *This solution is unique if and only if the transformation S of Theorem 9 has an inverse and in this case* $\mathfrak{N} \ominus \mathfrak{P} = \mathfrak{N}$.

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UNSTABLE MINIMAL SURFACES OF HIGHER TOPOLOGICAL TYPES

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Communicated October 24, 1940

1. We are ultimately concerned with extending the calculus of variations in the large to multiple integrals. The problem of the existence of minimal surfaces of unstable type contains many of the typical difficulties, especially those of a topological nature. In the case of m contours ($m > 1$) new difficulties appear not found either in the general theory when $m = 1$ or in the extensive minimum theory when $m > 1$. The case $m = 2$, however, appeared to contain the essentially new difficulties, and for simplicity we present this case, leaving the extensions for later papers.

2. The Curves g_0 and g_1.—We shall be concerned with two simple closed rectifiable curves g_0 and g_1 in n-space separated by an $(n - 1)$-plane and satisfying the chord arc condition. Under the chord arc condition the ratio of an arbitrary chord length to the lesser of the two corresponding arc lengths shall be bounded from 0. In certain aspects of our work not all these hypotheses are needed. For example, in the homotopy theorem the curves g_0 and g_1 need only be simple and closed.

3. Canonical Pairs (φ, h).—The curves g_k, $k = 0, 1$, shall each be of the form $x = g(t)$, in vector notation, where $0 \leqq t \leqq 2\pi$ with t proportional to the arc length and $g(t)$ with a period 2π in t. We shall use other representations of g of the form $g(h(\alpha))$ where $h(\alpha)$ is a non-decreasing transformation of α with the property that $h(\alpha + 2\pi) \equiv h(\alpha) + 2\pi$. The case where the discontinuities of $h(\alpha)$ occur exclusively at a set of points α which are congruent mod 2π to a constant c is called *degenerate*. Let $h(\alpha)$ and $k(\alpha)$ be two transformations and r any rational integer. The minimum of the Lebesgue integral

$$\frac{1}{2\pi}\int_0^{2\pi} (h(\alpha) - k(\alpha) + 2\pi r)^2 d\alpha$$

with respect to r will be denoted by $[h, k]$. We regard $[h, k]$ as a distance in a metric space I, proving that $[h, k]$ satisfies the triangle axiom and identifying transformations h and k if $[h, k] = 0$. The space I is compact.

We choose three constants $\alpha_1 < \alpha_2 < \alpha_3$ on the interval $0 \leqq \alpha < 2\pi$. Continuous transformations $\varphi(\alpha)$ which satisfy the condition $\varphi(\alpha_i) = \alpha_i$ are termed *restricted* and form a subspace I^* of I. Such transformations will be denoted by Greek letters. A directly conformal 1 to 1 transformation of the disc $u^2 + v^2 \leqq 1$ into itself induces a transformation $T(\theta)$ of the point $(1, \theta)$ on the circle $r = 1$ into a point $(1, T(\theta))$. We term $T(\theta)$ a *Mobius*

transformation. A restricted transformation $\varphi(\alpha)$ and a general transformation $h(\alpha)$ such that $\varphi(\alpha) \equiv h(T(\alpha))$ are said to be *connected* by T. A Mobius transformation T connecting a restricted transformation φ with a continuous transformation h is uniquely determined on I by φ and h, and maps a subspace of $I^* \times I$ continuously onto a subspace of I. The space of Mobius transformations completed with the degenerate transformations is a compact subspace of I. Pairs (φ, h) are termed *canonical* if φ is restricted, and if h is either connected with φ or is degenerate. The limit (φ, h) on I^2 of a sequence of canonical pairs is canonical whenever φ is continuous.

4. *The Dirichlet Sum D.*—Corresponding to a vector $p(\theta)$ defining a simple closed curve, we introduce the Douglas functional

$$A(p) = \frac{1}{16\pi} \int_G \int \frac{[p(\alpha) - p(\beta)]^2}{\sin^2 \dfrac{(\alpha - \beta)}{2}} d\alpha \, d\beta \qquad (1)$$

with G the parallelogram $(0 \leqq \alpha \leqq 2\pi)$, $(-\pi \leqq \alpha - \beta \leqq \pi)$.

Let B be a ring region bounded by a circle C_0: $r = 1$ and a circle C_1: $r = \rho$, $0 < \rho < 1$. Let $H(u, v)$ be the harmonic surface in vector form defined over B with the respective boundary values

$$p_k(\theta) = g_k[h_k(\theta)] \quad (k = 0, 1) \qquad (2)$$

where $h_k(\theta)$ is a continuous transformation. The usual Dirichlet integral sum for $H(u, v)$ will be denoted by $D(p_0, p_1, \rho)$. Its value [6] is

$$D(p_0, p_1, \rho) = A(p_0) + A(p_1) + R(p_0, p_1, \rho) \qquad (0 < \rho < 1) \quad (3)$$

where R is given in [2], p. 280. We define $D(p_0, p_1, \rho)$ when $\rho = 0$ or when h_0 or h_1 is degenerate, by (3), noting that R is defined in these cases in [2].

The function D and space of sets (p_0, p_1, ρ) is inadequate because the degenerate points offer trivial minima, the space is not simply connected, D is not weakly upper reducible (defined in [1]) at degenerate points, and because the representation of minimal surfaces by critical sets is needlessly multiple, thereby complicating the topology.

5. *The Space* Π *and function* $W(P)$.—A product of metric spaces will here be metricized by a distance function which is the sum of the component distance functions. Let J denote the interval $0 \leqq \rho < 1$. The space Π shall consist of sets or points $P = (\varphi_0, \varphi_1, h_0, h_1, \rho)$ in which ρ varies on J, $h(0) = 0$, and (φ_k, h_k), $k = 0, 1$, is a canonical pair of transformations. The pair $\{\varphi_0, \varphi_1\}$ will be termed the *restricted projection* of P and will be regarded as a point on I^2. When $\rho = 0$, two points P with the same restricted projection are identified. When $\rho > 0$ points P are identified as on $I^4 J$. Let $d(P, Q)$ be the distance between P and G as points of $I^4 J$. With $Q = (\psi_0, \psi_1, k_0, k_1, \sigma)$ let

$$\delta(P, Q) = [\varphi_0 \psi_0] + [\varphi_1 \psi_1] + \rho + \sigma.$$

The points P form a space Π on which our final choice of distance function PQ shall be the minimum of $d(P, Q)$ and $\delta(P, Q)$. The usual axioms are satisfied by PQ.

Set

$$g_k(\theta) = g_k[\varphi_k(\theta)] \qquad A(g_k, \varphi_k) = A(g_k).$$

To define $W(P)$ we use (2) and set

$$W(P) = A(g_0, \varphi_0) + A(g_1, \varphi_1) + R(p_0, p_1, \rho).$$

As one sees $W(P) > D$ when h_0 or h_1 is degenerate. We show that $W(P)$ is regular at infinity, and boundedly compact in the sense of [1]. Moreover, the connectivity R_0 of the subspace on which W is finite is 1, and the other connectivities are 0. A point P at which $\rho > 0$ is termed *differentially critical* when the boundary vectors (2) define a ring minimal surface over B; or, if $\rho = 0$, when these vectors define disc minimal surfaces. The homotopy theorem states that a homotopic critical point is differentially critical.

We show that $W(P)$ is weakly upper reducible at each ordinary or degenerate point, and if $g'_k(t)$ satisfies a Lipschitz condition at critical points. *The general theory thus applies with the consequent relations between the critical sets classified as to type members.* The following special theorem is provable by a limiting process [11] under the original weak hypothesis. These hypotheses are weaker than those of Shiffman [4] in the case of one contour.

THEOREM. *If g_1 and g_1 bound a ring minimal surface belonging to a minimizing set of critical points of W, there either exists a ring minimal surface of non-minimizing type bounded by g_0 and g_1, or else a disc minimal surface of non-minimizing type bounded by g_0 or by g_1.*

If g_0 and g_1 possess convex plane projections, the first alternative of the theorem alone holds.

When $\rho = 0$, W is the sum of the functions $A(g_k, \varphi_k)$ defined on the product space $I^* \times I^*$. Critical points in this subspace correspond to two disc minimal surfaces. Let a function $f(x) + f(y) = F$ be defined on a product of two metric spaces of points x and y, respectively. Are the type numbers of critical points of F obtainable by summation from those of $f(x)$ and $f(y)$ in the natural way, that is, as in the case where $f(x)$ and $f(y)$ are non-degenerate forms? This question has been answered affirmatively by A. E. Pitcher under conditions which admit a wide range of applications. Pitcher's result will presently be published.

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